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Tomul XXXVI (XL)

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Secția I

**MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

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Secția I

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Section I

MATHEMATICS. THEORETICAL MECHANICS. PHYSICS

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BY

VALERIU MUNTEANU, M. LUCANU and ADRIANA PĂSTRĂVANU

In order to correct the errors occurred in a received word, one first calculates its syndrom. If from the syndrom structure one can deduce that in the received word there are k errors, than the respective syndrom is denoted by $[Z_k]$. One than successively errs the received word's symbols. If after erring a symbol in the received word the corresponding syndrom becomes $[Z_{k+1}]$, it follows that the respective symbol was correctly received, and if the syndrom becomes $[Z_{k-1}]$, it follows that the respective symbol was erroneously received.

[illegible]
$$(2) \begin{bmatrix} 1 & \alpha & \alpha^3 & \alpha^{n-1} \\ 1 & \alpha^3 & (\alpha^3)^2 & (\alpha^3)^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{2^{t-1}} & (\alpha^{2^{t-1}})^2 & (\alpha^{2^{t-1}})^{n-1} \end{bmatrix} \begin{bmatrix} a'_0 \\ a'_1 \\ \vdots \\ a'_{n-1} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_3 \\ \vdots \\ S_{2^{t-1}} \end{bmatrix}$$
$$(3) \quad \left. \begin{aligned} S_3 &= S_1^2, \\ S_4 &= S_2^2, \\ S_6 &= S_3^2, \\ &\vdots \end{aligned} \right\}$$

With a view to deriving the number of linear independent equations in the system (1) it is necessary to determine the rank of the matrix

$$(4) \quad \mathcal{M}_e = \begin{bmatrix} S_e & S_{e-1} & \dots & S_1 \\ S_{e+1} & S_e & \dots & S_2 \\ \dots & \dots & \dots & \dots \\ S_{2e-1} & S_{2e-2} & \dots & S_e \end{bmatrix}.$$

In [2] it is shown that if the rank of the matrix is k , $k \leq e-1$, than all the errors in the received word can be corrected by means of the following calculation stages:

a) if $\det \mathcal{M}_k \neq 0$, $k \leq e-1$, one decides that during the transmission, k errors have occurred;

b) one successively errs the symbols of the received word and if the rank of the matrix becomes $k+1$, one decides that the respective symbol was correctly received, and if the rank of the matrix becomes $k-1$, one decides that the respective symbol was erroneously received.

3. The Proposed Decoding Procedure. In [1] it was shown that in the case of the BCH binary codes, the Newton identities hold

$$(5) \quad \begin{cases} S_1 \oplus \sigma_1 = 0, \\ S_3 \oplus \sigma_1 S_2 \oplus \sigma_2 S_1 \oplus \sigma_3 = 0, \\ S_5 \oplus \sigma_1 S_4 \oplus \sigma_2 S_3 \oplus \sigma_3 S_2 = 0, \\ \dots \\ S_{2e-1} \oplus \sigma_1 S_{2e-2} \oplus \sigma_2 S_{2e-3} \oplus \dots \oplus \sigma_e S_{e-1} = 0. \end{cases}$$

It is also proved in [1] that the determinant of the system (5)

$$(6) \quad \det \mathcal{M}_e = \begin{vmatrix} 1 & 0 & 0 \dots 0 \\ S_2 & S_1 & 1 \dots 0 \\ S_4 & S_3 & S_2 \dots 0 \\ \dots & \dots & \dots \\ S_{2e-2} & S_{2e-3} & S_{2e-4} \dots S_{e-1} \end{vmatrix} = \begin{vmatrix} S_1 & 1 & 0 & \dots & 0 \\ S_3 & S_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ S_{2e-3} & S_{2e-4} & S_{2e-5} & \dots & S_{e-1} \end{vmatrix}$$

is nonzero if and only if during the transmission e or $e-1$ errors have occurred.

Taking this property into consideration, the decoding procedure proposed in this paper consists of the following steps:

a) It is supposed that during the transmission there can occur $e-1$ errors at most.

b) By means of the relation (2) one calculates the amounts $S_1, S_3, \dots, S_{2e-1}$ and by means of (3), the amounts $S_2, S_4, \dots, S_{2e-2}$.

c) One determines the rank of the matrix $\mathcal{M}_e$ given by the relation (4); if the calculated rank is k , one decides that k errors have appeared during the transmission.

d) One successively errs the symbols of the received word; if after a symbol was deliberately erred the determinant of the system formed from the first $k+1$ equations retained from (5) results nonzero, one decides that the respective symbol was correctly received and if the respective determinant results null, one decides that the respective symbol was erroneously received.

4. Example. Let us suppose that through a transmission channel one transmits code words of length $n=15$ and it is known that there can occur two errors at most. Receiving the word $v'(X)=1\oplus X\oplus X^6\oplus X^9\oplus X^{10}\oplus X^{11}\oplus X^{12}\oplus X^{14}$, the question is to determine the polynomial associated with the transmitted code word.

Solution: Since the channel can cause two errors at most and $n=15$, it results that the roots of the code words must be α , α^3 and α^5 from $GF(2^4)$, where α is a primitive element of this field.

The parity-check matrix in this particular case is of the form

$$(7) \quad [H] = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 & \alpha^7 & \alpha^8 & \alpha^9 & \alpha^{10} & \alpha^{11} & \alpha^{12} & \alpha^{13} & \alpha^{14} \\ 1 & \alpha^3 & \alpha^6 & \alpha^9 & \alpha^{12} & 1 & \alpha^3 & \alpha^6 & \alpha^9 & \alpha^{12} & 1 & \alpha^3 & \alpha^6 & \alpha^9 & \alpha^{12} \\ 1 & \alpha^5 & \alpha^{10} & 1 & \alpha^5 & \alpha^{10} & 1 & \alpha^5 & \alpha^{10} & 1 & \alpha^5 & \alpha^{10} & 1 & \alpha^5 & \alpha^{10} \end{bmatrix}.$$

By means of the relation (2) it results $S_1=\alpha^9$, $S_3=\alpha^2$ and $S_5=0$ and by means of the relation (3) it results $S_2=\alpha^3$ and $S_4=\alpha^6$.

From (4) it follows that

$$(8) \quad \det \mathcal{M}_3 = \begin{vmatrix} \alpha & \alpha^3 & \alpha^9 \\ \alpha & \alpha^2 & \alpha^3 \\ 0 & \alpha^6 & \alpha^2 \end{vmatrix} = \alpha^2(\alpha^4 \oplus \alpha^9) \oplus \alpha^6(\alpha^5 \oplus \alpha^{15}) = \alpha^2\alpha^{14} \oplus \alpha^9\alpha^{10} = 0.$$

This implies that during the transmission there have occurred less than three errors.

Calculating

$$(9) \quad \det \mathcal{M}_2 = \begin{vmatrix} \alpha^2 & \alpha^3 \\ \alpha^6 & \alpha^2 \end{vmatrix} = \alpha^4 \oplus \alpha^9 \neq 0,$$

one can deduce that in the received word there are two errors.

The determinant from the relation (6) becomes

$$(10) \quad \det \mathcal{M}_3 = \begin{vmatrix} 1 & 0 & 0 \\ S_2 & S_1 & 1 \\ S_4 & S_3 & S_2 \end{vmatrix} = \begin{vmatrix} S_1 & 1 \\ S_3 & S_2 \end{vmatrix}.$$

Erring successively the symbols of the received word and calculating every time the amounts S_1 , S_2 , S_3 , it follows after the calculations that only when errors appear in the third and twelfth position it results

$$(11) \quad \det \mathcal{M}_2^{(3)} = \begin{vmatrix} \alpha^{11} & 1 \\ \alpha^3 & \alpha^7 \end{vmatrix} = \alpha^{18} \oplus \alpha^3 = \alpha^3 \oplus \alpha^3 = 0,$$

respectively

$$(12) \quad \det \mathcal{M}_3^{(12)} = \begin{vmatrix} \alpha^2 & 1 \\ \alpha^6 & \alpha^4 \end{vmatrix} = \alpha^6 \oplus \alpha^6 = 0$$

all the others determinants $\det \mathcal{M}_3^{(i)} \neq 0$, for $i=1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14$.

From the exposed theory and the performed calculations it follows that there have occurred errors on the third and twelfth position, and so, the transmitted code word corresponds to the polynomial $v(X) = 1 \oplus X \oplus X^2 \oplus X^6 \oplus X^9 \oplus X^{10} \oplus X^{12} \oplus X^{14}$.

5. Conclusions. 1. In order to emphasize the superiority of the decoding method for the BCH codes exposed in this paper, a comparison can be made between the methods exposed in [1] and [2] and the present method.

2. In the method proposed in [1] for the decoding there are necessary the following steps:

a) by means of (2) one calculates the amounts $S_1, S_3, \dots, S_{2e-1}$ and by means of (3) the amounts $S_2, S_4, \dots, S_{2e-2}$;

b) one determines the amounts $\sigma_1, \sigma_2, \dots, \sigma_e$ from the system (5). During this step, a certain number of trials are performed because correct solutions are obtained only for the case when the number of equations used from the system (5) equals the number of errors or the number of errors minus one;

c) one substitutes then the elements of the field $GF(2^m)$ in the polynomial

$$P(X) = X^e \oplus \sigma_1 X^{e-1} \oplus \dots \oplus \sigma_e.$$

The elements which satisfy the identity indicate the positions of the errors.

3. In comparison with this method, in the proposed procedure the b) and c) steps which necessitate the greatest volume of calculations are no more necessary. This two steps are reduced only to the determination of the rank or some matrices, without any trials for the establishment of the number of errors which have effectively occurred during the transmission.

Using the method proposed in [2] for the decoding the following steps are necessary:

a) by means of the relation (2) one calculates the amounts $S_1, S_3, \dots, S_{2e-1}$ and by means of (3), the amounts $S_2, S_4, \dots, S_{2e-2}$;

b) one determines the rank of the matrix $\mathcal{M}_e$ given by the relation (4), so establishing the number of errors which have effectively occurred during the transmission; let us suppose that this number is $k \leq e-1$;

c) one successively errs the symbols of the received word, calculating every time the rank of the system matrix. If this equals $k-1$, the respective symbol was erred and if the rank is $k+1$, the respective symbol was correctly received.

4. Comparing the structure of the matrix $\mathcal{M}_e$ given by (4) with the structure of the matrix given by (6) it can be observed that the amount of calculation for the rank of the matrix given by (6) is less than for matrix given by (4). All these certify the superiority of the decoding method proposed in the present paper in comparison with that given in [2].

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O NOUĂ METODĂ DE DECODARE A CODURILOR BCH BINARE

(Rezumat)

Se expune o nouă metodă de decodare a codurilor BCH binare bazată pe identitățile lui Newton. Metoda propusă este superioară celor cunoscute în literatura de specialitate datorită faptului că necesită un volum de calcul mai mic.

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UNE SOLUTION DES ÉQUATIONS GRAVITATIONALES D'EINSTEIN

PAR

ELENA PUȘCAȘU

1. Introduction. Soit

$$(1) \quad R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \chi T_{\alpha\beta}, \quad (\alpha, \beta = 0, 1, 2, 3)$$

les équations gravitationales d'Einstein de la théorie de relativité, où : $g_{\alpha\beta}$ est le tenseur métrique d'un espace Riemann avec 4 dimensions V_4 , $R_{\alpha\beta}$ est le tenseur Ricci de cet espace, R — l'invariant Ricci, $T_{\alpha\beta}$ — le tenseur énergie-impulsion et χ une constante universelle.

Le tenseur $T_{\alpha\beta}$ satisfait la condition de conservation

$$(2) \quad \nabla_\alpha T_\beta^\alpha = 0.$$

En supposant le tenseur $T_{\alpha\beta}$ connu, (1) représente un système d'équations avec dérivées partielles de deuxième ordre aux inconnues $g_{\alpha\beta}$ et, le principal problème de la théorie de relativité est de déterminer le tenseur $g_{\alpha\beta}$. Une fois déterminée $g_{\alpha\beta}$, est connue la métrique ds^2 de l'espace V_4 . Le système (1) est, en général, difficile à résoudre et on connaît les solutions de cet système seulement dans des cas spéciaux [1], [2], [4]. Dans cette Note on donne une solution des équations d'Einstein et on étudie quelques propriétés géométriques de cette solution.

2. Une solution des équations d'Einstein. Nous supposons la métrique de l'espace V_4 statique et avec une symétrie sphérique. Dans des coordonnées polaires elle a la forme

$$(3) \quad ds^2 = e^{2n} c^2 dt^2 - e^{2l} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

où n et l sont des fonctions seulement de r [3].

Pour la métrique (3) les équations d'Einstein ont la forme :

$$(4) \quad \begin{aligned} \frac{2l'}{r} - \frac{1}{r^2} + \frac{1}{r^2} e^{2l} &= T_{00} e^{2l-2n}, \\ \frac{2n'}{r} + \frac{1}{r^2} - \frac{1}{r^2} e^{2l} &= \chi T_{11}, \end{aligned}$$

$$R^2 e^{-2l} (n' + n'^2 - l'n' + \frac{n' - l'}{r}) = \chi T_{22}.$$

Le tenseur énergie-impulsion $T_{\alpha\beta}$, englobe un terme d'origine électromagnétique représenté par le tenseur de Maxwell $\tau_{\alpha\beta}$ et un terme qui donne la distribution de masse représenté par le tenseur matériel $M_{\alpha\beta}$.

Nous avons

$$(5) \quad T_{\alpha\beta} = \tau_{\alpha\beta} + M_{\alpha\beta}.$$

Pour la métrique (3) le tenseur de Maxwell possède les composantes :

$$(6) \quad \tau_{00} = \frac{h^2}{2r^4} e^{-2l}, \quad \tau_{11} = \frac{h^2}{2r^4} e^{-2n}, \quad \tau_{22} = \frac{h^2}{2r^2} e^{-2n-2l}, \quad \tau_{33} = \tau_{22} \sin^2 \theta,$$

où h est une constante que caractérise la particule qui crée le champs électromagnétique [2], [3].

En suite, nous nous proposons à déterminer l'espace V_4 dans les conditions suivantes :

$$(7) \quad M_{\alpha\beta} = \lambda \tau_{\alpha\beta} \quad (\lambda \text{ étant une constante arbitraire}).$$

De (5), (6) et (7) il en résulte

$$(8) \quad T_{00} = (1+\lambda) \frac{h^2}{2r^4} e^{-2l}, \quad T_{11} = -(1+\lambda) \frac{h^2}{2r^4} e^{-2n},$$

$$T_{22} = (1+\lambda) \frac{h^2}{2r^2} e^{-2n-2l}, \quad T_{33} = T_{22} \sin^2 \theta.$$

Pour $T_{\alpha\beta}$ donné par (8), de (4) on obtient

$$(9) \quad e^{2n} = 1 + \frac{a}{r} + \frac{b}{r^2}, \quad e^{2l} = e^{-2n},$$

où a est une constante d'intégration et

$$(10) \quad b = \frac{(1+\lambda)\chi h^2}{2}.$$

Donc, l'espace V_4 a la métrique

$$(11) \quad ds^2 = \left(1 + \frac{a}{r} + \frac{b}{r^2}\right) c^2 dt^2 - \frac{1}{1 + \frac{a}{r} + \frac{b}{r^2}} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

Remarque. Pour $b=0$ ($\lambda=-1$) nous avons $T_{\alpha\beta}=0$ et (11) donne la métrique de Schwarzschild [1]. De suite, nous supposons $b \neq 0$.

3. Propriétés géométriques de la solution (11).

P_1 . La métrique donnée par (11) est une métrique régulière.

Vraiment, pour $r \rightarrow \infty$, (11) donne la métrique de Minkovski.

P_2 . La métrique donnée par (11) est une métrique Reissner-Weyl.

Les composantes du tenseur Ricci pour la métrique (11) sont données par

$$(12) \quad R_{00} = \frac{b}{r^4} g_{00}, \quad R_{11} = \frac{b}{r^4} g_{11}, \quad R_{22} = -\frac{b}{r^4} g_{22}, \quad R_{33} = R_{22} \sin^2 \theta.$$

Donc

$$(13) \quad R = R_{\alpha\beta} g^{\alpha\beta} = 0.$$

P_3 . La distribution donnée par le tenseur $T_{\alpha\beta}$ est normale.

Les valeurs propres du tenseur $T_{\alpha\beta}$ sont données par

$$(14) \quad s_0 = s_1 = \frac{b}{\chi r^4}, \quad s_2 = s_3 = -\frac{b}{\chi r^4},$$

et les vecteurs propres correspondants sont donnés par

$$(15) \quad \begin{aligned} u(u_\alpha), \quad u_\alpha &= (\sqrt{2}e^n, e^t, 0, 0), \\ v(v_{1\alpha}), \quad v_{1\alpha} &= e^n, \sqrt{2}e^t, 0, 0), \\ v_2(v_{2\alpha}), \quad v_{2\alpha} &= (0, 0, r \sin \theta, -r \sin \theta \cos \theta), \\ v_3(v_{3\alpha}), \quad v_{3\alpha} &= (0, 0, -r \cos \theta, -r \sin^2 \theta). \end{aligned}$$

On montre que les vecteurs (15) sont perpendiculaires et

$$(16) \quad ||u||^2 = 1, \quad ||v_1||^2 = ||v_2||^2 = ||v_3||^2 = -1.$$

Donc, le vecteur u est orienté en temps et les vecteurs v_1, v_2, v_3 sont orientés en espace.

Parce que le tenseur $T_{\alpha\beta}$ a un vecteur propre orienté en temps, la distribution donnée par $T_{\alpha\beta}$ est normale.

Les vecteurs u, v_1, v_2, v_3 déterminent le repère principal associé au tenseur $T_{\alpha\beta}$.

Les valeurs propres du tenseur $T_{\alpha\beta}$ ont les suivantes interprétations physiques :

$s_0 = \rho$ = la densité de la distribution matérielle ;

$$(17) \quad \begin{aligned} s_1 &= -p_1, \quad s_2 = -p_2, \quad s_3 = -p_3; \\ p_1, p_2, p_3 &\text{ sont les pressions partielles.} \end{aligned}$$

En rapport avec le repère principal, $T_{\alpha\beta}$ est donné par

$$(18) \quad T_{\alpha\beta} = \rho u_\alpha u_\beta + p_1 v_{1\alpha} v_{1\beta} + p_2 v_{2\alpha} v_{2\beta} + p_3 v_{3\alpha} v_{3\beta}.$$

De (14) et (17) résulte

$$(19) \quad p_1 + p_2 + p_3 = \rho.$$

Parce que $b \neq 0$, les pressions partielles ne peuvent pas être simultanées nulles, donc la distribution matérielle ne peut pas être pure.

P_4 . L'espace physique associé à l'espace V_4 admet un groupe de mouvements G à trois paramètres.

Vraiment, l'espace physique F au moment t , associé à l'espace V_4 a la métrique

$$(20) \quad d\sigma^2 = \frac{r^2}{r^2 + ar + b} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

On montre que l'espace F admet trois vecteurs Killing linéaires indépendants ξ_1, ξ_2, ξ_3 qui sont donnés par

$$\begin{aligned}\xi_1 &= (0, \cos \varphi, -\sin \varphi \operatorname{ctg} \theta), \\ (21) \quad \xi_2 &= (0, \sin \varphi, \cos \varphi \operatorname{ctg} \theta), \\ \xi_3 &= (0, 0, 1),\end{aligned}$$

donc, F admet un groupe de mouvements G à trois paramètres.

P_1 . Le groupe G est parfait.

Si nous notons par X_1, X_2, X_3 les opérateurs infinitésimales du groupe G , les équations de structure du groupe G sont données par $(X_1, X_2) = -X_3$, $(X_1, X_3) = X_2$, $(X_2, X_3) = -X_1$, d'où résulte que G est un groupe parfait.

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O SOLUȚIE A ECUAȚIILOR GRAVITAȚIONALE ALE LUI EINSTEIN

(Rezumat)

Se determină o soluție a ecuațiilor gravitaționale ale lui Einstein pentru o anumită distribuție de masă și energie. În continuare se studiază câteva proprietăți geometrice și fizice ale acestei soluții.

ELLIPSOIDS AND REGULAR MAPPINGS IN RIEMANNIAN MANIFOLDS

BY

VERONICA T. BORCEA and AL. NEAGU

The aim of this Note is to prove that a homeomorphism between two infinite dimensional Riemannian manifolds is regular iff its linear tangent map is regular.

Let H be a Hilbert space with the scalar product $\langle, \rangle$. We define an ellipsoid E in H as the image of a sphere $S(X_0, t)$ by an affine transformation $A: H \rightarrow H$, $Y = AX$, where $AX = TX + a$, with $T: H \rightarrow H$ linear, continuous, bijective operator and $a \in H$.

If the sphere $S(X_0, t)$ has the equation $\langle X - X_0, X - X_0 \rangle = t^2$, then the equation of ellipsoid $E_t(G, Y_0) = A(S(X_0, t))$, $Y_0 = AX_0$, will be $\langle G(Y - Y_0), Y - Y_0 \rangle = t^2$, where $G = (T^{-1})^* T^{-1}$. Clearly G is a self-adjoint, positively defined operator, called the characteristic of the ellipsoid $E_t(G, Y_0)$.

The extreme semiaxes of $E_t(G, Y_0)$ are

$$(1) \quad a_1(t, X_0) = \sup_{\|X - X_0\| = t} \|AX - AX_0\|; \quad a_0(t, X_0) = \inf_{\|X - X_0\| = t} \|AX - AX_0\|.$$

Remark 1. We have

$$(2) \quad a_1(t, X_0) = \sup \{ \|T(X - X_0)\|; \|X - X_0\| = t \} = t \|T\|,$$

$$(3) \quad a_0(t, X_0) = \inf \{ \|T(X - X_0)\|; \|X - X_0\| = t \} = t \|T^{-1}\|^{-1}.$$

Definition 1. We call the *principal characteristic parameter* of the ellipsoid $E_t(G, Y_0)$, the ratio

$$(4) \quad p_1 = \frac{a_1(t, X_0)}{a_0(t, X_0)}.$$

We also say that p_1 is the characteristic parameter of the affine transformation A .

Remark 2. By (2), (3) and (4) it follows

$$(5) \quad p_1 = \|T\| \|T^{-1}\|.$$

Let V be a Riemannian manifold and its model the Hilbert space H . We denote by d the geodesic metric, TV the tangent bundle, $T_x V$ the tangent space at $x \in V$ and by $\langle, \rangle_x$ (or simply $\langle, \rangle$) the Riemannian metric. We consider an automorphism $\Phi: TV \rightarrow TV$, with the restriction $\Phi_x: T_x V \rightarrow T_x V$ and

Φ^* the adjoint of Φ . The restriction $G_x = \Phi_x^{*-1} \Phi_x^{-1} : T_x V \rightarrow T_x V$ of the automorphism $G = \Phi^{*-1} \Phi^{-1} : TV \rightarrow TV$, is a self-adjoint operator, positively defined. In every tangent space $T_x V$, we define the family of homotetical ellipsoids $E_t(G_x) = \{E_t(G_x); t \in (0, \infty)\}$, where $E_t(G_x) = \{X; X \in T_x V, \langle G_x X, X \rangle = t^2\}$.

Because in the finite dimensional case, the operator G_x determines completely the characteristics of the ellipsoids $E_t(G_x)$, we call G the *field of characteristics* on V and $E_t(G) = \{E_t(G_x); x \in V\}$ the *field of ellipsoids*.

For t sufficiently small, the image $\mathcal{E}_t(G_x)$ of $E_t(G_x)$ by $\exp_x$ is called *geodesic ellipsoid* centred at $x \in V$. Analogously we obtain the *field of geodesic ellipsoids* $\mathcal{E}_t(G)$.

Remark 3. $E_t(G_x)$ and $\mathcal{E}_t(G_x)$ have the same characteristics.

Let $f : V \rightarrow \tilde{V}$ be a differentiable mapping ($V, \tilde{V}$ are Riemannian manifolds over the Hilbert spaces H , respectively $\tilde{H}$). The map f is called a *local isomorphism* at $x \in V$ if $T_x f$ is bijective ($T_x f$ denote the linear tangent mapping). The point x is called in this case *A-point*.

We consider a homeomorphism $\varphi : B(0, 1) \rightarrow V$, $\varphi(0) = x_0 \in V$, where $B(0, 1) \subset H$ is the unit ball, and we denote by $\sigma_\alpha = \varphi(S(0, \alpha))$, $\alpha \in (0, 1)$. The hypersurface family $\{\sigma_\alpha\}$ is said to be *regular of parameter* $k = k(x_0)$, relative to a point x_0 , if

$$(6) \quad \limsup_{\alpha \rightarrow 0} \frac{\sup \{d(x, x_0); s \in \sigma_\alpha\}}{\inf \{d(x, x_0); x \in \sigma_\alpha\}} = k(x_0) < \infty.$$

Definition 2. A homeomorphism $f : V \rightarrow \tilde{V}$ is called *regular* at $x_0 \in V$, if it maps a regular family of parameter $k(x_0)$, relatively to x_0 , into a regular family of a parameter $\tilde{k}(x_0)$, relatively to $f(x_0)$. The homeomorphism f is said to be *regular* in V if it is regular at each of its points.

The function $q(x) = \inf k(x)$, $\tilde{k}(x)$, where the infimum is taken over all regular families $\{\sigma_\alpha\}$ relatively to x , called the *characteristic* of the mapping f .

Remark 4. If $\{\sigma_\alpha\}$ is a regular hypersurface family of parameter k relatively to x , then the family $\{\tau_\alpha\}$, $\tau_\alpha = \exp_x^{-1}(\sigma_\alpha)$, is a regular hypersurface family of the same parameter k , respectively to $0_x \in T_x V$.

Proposition 1. If $A : H \rightarrow \tilde{H}$ is an affine transformation, then A is a regular transformation and its principal characteristic parameter $p_1(x)$ coincides with its characteristic $q(x)$.

Proof. Let $A : H \rightarrow \tilde{H}$ be, $\tilde{X} = AX = UX + a$, where U is a continuous linear, bijective operator, $a \in \tilde{H}$. The ellipsoid $E_t(G, X_0)$, $G = U^*U$ is mapped by A into a sphere $S(X_0, t)$. The hypersurface family $\{\sigma_t\} = \{E_t(G, X_0), t \in (0, 1)\}$ is clearly a regular family relative to X_0 with $k(X_0) = p_1(X_0) = a_1(t, X_0)/a_0(t, X_0) = \|U\| \|U^{-1}\|$. Because $A(\sigma_t)$ is sphere for every $t \in (0, 1)$, it follows that $\tilde{k}(X_0) = 1$ and so $k(X_0) \cdot \tilde{k}(X_0) = p_1(X_0)$. It follows that

$$(7) \quad q(X_0) = \inf k(X_0) \tilde{k}(X_0) \leq p_1(X_0),$$

where the infimum is taken over all regular families $\{\sigma_\alpha\}$ relatively to X_0 .

If we consider the regular family $\{\sigma_\alpha\}$ of parameter $k(X_0) \geq p_1(X_0)$, we have $k(X_0) \tilde{k}(X_0) \geq p_1(X_0)$ and so $q(X_0) \geq p_1(X_0)$.

Now, let us consider a regular family $\{\bar{\sigma}_\alpha\}$ of parameter $\bar{k}(X_0) < p_1(X_0)$, and the hypersurface $\bar{\sigma}_\alpha \subset B(X_0, r) \setminus B(X_0, r_0)$, where :

$$r_0 = \inf \{ \|X - X_0\| ; X \in \bar{\sigma}_\alpha \}, \quad r_1 = \sup \{ \|X - X_0\| ; X \in \bar{\sigma}_\alpha \}.$$

Let us consider the ellipsoids $E_{t_0}(G, X_0)$ and $E_{t_1}(G, X_0)$ with

$$a_0(t_0) = r_0 = t_0 \|U^{-1}\|, \quad a_1(t_1) = r_1 = t_1 \|U^{-1}\|.$$

Clearly

$$\bar{k}(X_0) = \frac{r_1}{r_0} = \frac{a_1(t_1)}{a_0(t_0)} = \frac{t_1}{t_0} p_1(X_0).$$

Since $\bar{k}(X_0) < p_1(X_0)$, we have $t_1 < t_0$. The images of the ellipsoids $E_{t_0}(G, X_0)$ and $E_{t_1}(G, X_0)$ by A are the spheres $S(\tilde{X}_0, t_0)$ and respectively $S(\tilde{X}_0, t_1)$. It follows that

$$t_0 \leq \sup \{ \|AX - X_0\| ; X \in \bar{\sigma}_\alpha \}, \quad t_1 \geq \inf \{ \|AX - X_0\| ; X \in \bar{\sigma}_\alpha \},$$

and so

$$\tilde{k}(X_0) \geq \frac{t_0}{t_1} = \frac{p_1(X_0)}{\bar{k}(X_0)}.$$

We obtain $\tilde{k}(X_0) \cdot k(X_0) \geq p_1(X_0)$, hence $q(X_0) \geq p_1(X_0)$ and, since $k(X_0) \cdot \tilde{k}(X_0) = p_1(X_0)$ for the family $\{\sigma_t\}$, it follows that $q(X) = p_1(X)$ as desired.

COROLLARY. If $\tilde{p}_1$ is the principal characteristic parameter of the ellipsoids, images of spheres by the affine transformation A , then $\tilde{p}_1(X) = q(X)$.

PROPOSITION 2. Let $F: H \rightarrow \tilde{H}$ be a homeomorphism with $F(0) = 0$, for which 0 is an A -point. F maps a regular hypersurface family $\{\tau_\alpha\}$ of parameter k , relatively to 0, into a regular hypersurface family of parameter $\tilde{k}$ iff $DF(0)$ maps the family $\{\tau_\alpha\}$ into a regular family of parameter $\tilde{k}$.

PROOF. For the sufficiency we consider the regular family $\{\tau_\alpha\}$ of parameter k relatively to 0, hence

$$(8) \quad \frac{\sup \{ \|X\| ; X \in \tau_\alpha \}}{\inf \{ \|X\| ; X \in \tau_\alpha \}} = k + \varepsilon_\alpha, \quad \text{where} \quad \lim_{\alpha \rightarrow 0} \varepsilon_\alpha = 0.$$

We denote by

$$(9) \quad \tilde{k}^* = \limsup_{\alpha \rightarrow 0} \frac{\sup \{ \|F(X)\| ; X \in \tau_\alpha \}}{\inf \{ \|F(X)\| ; X \in \tau_\alpha \}}.$$

Since

$$(10) \quad F(X) = DF(0)X + \varepsilon(0, X)\|X\|, \quad \text{with} \quad \lim_{\|X\| \rightarrow 0} \|\varepsilon(0, X)\| = 0,$$

we obtain

$$\limsup_{\alpha \rightarrow 0} \frac{\sup_{X \in \tau_\alpha} \|DF(0)X\| - \sup_{X \in \tau_\alpha} \|X\| \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\|}{\inf_{X \in \tau_\alpha} \|DF(0)X\| + \sup_{X \in \tau_\alpha} \|X\| \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\|} \leq \tilde{k}^* \leq$$

$$\leq \limsup_{\alpha \rightarrow 0} \frac{\sup_{X \in \tau_\alpha} \|DF(0)X\| + \sup_{X \in \tau_\alpha} \|X\| \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\|}{\inf_{X \in \tau_\alpha} \|DF(0)X\| - \sup_{X \in \tau_\alpha} \|X\| \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\|},$$

from which

$$1 - \frac{\sup_{X \in \tau_\alpha} \|X\|}{\sup_{X \in \tau_\alpha} \|DF(0)X\|} \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\|$$

$$\tilde{k} \limsup_{\alpha \rightarrow 0} \frac{1 + \inf_{X \in \tau_\alpha} \|X\|}{1 + \inf_{X \in \tau_\alpha} \|X\|} \frac{\sup_{X \in \tau_\alpha} \|X\|}{\inf_{X \in \tau_\alpha} \|DF(0)X\|} \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\| \leq \tilde{k}^* \leq$$

$$\frac{\sup_{X \in \tau_\alpha} \|X\|}{1 + \frac{\sup_{X \in \tau_\alpha} \|X\|}{\sup_{X \in \tau_\alpha} \|DF(0)X\|}} \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\|$$

$$\leq \tilde{k} \limsup_{\alpha \rightarrow 0} \frac{\sup_{X \in \tau_\alpha} \|X\|}{1 - \frac{\inf_{X \in \tau_\alpha} \|X\|}{\inf_{X \in \tau_\alpha} \|DF(0)X\|}} \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\|$$

Since $DF(0)$ is a linear, continuous, bijective operator it follows that

$$(11) \quad \frac{\|X\|}{\|(DF(0))^{-1}\|} \leq \|DF(0)X\| \leq \|DF(0)\| \|X\|,$$

and so

$$(12) \quad \frac{1}{\|DF(0)X\|} \leq \frac{\sup_{X \in \tau_\alpha} \|X\|}{\sup_{X \in \tau_\alpha} \|DF(0)X\|} \leq \|(DF(0))^{-1}\|,$$

$$(13) \quad \frac{1}{\|DF(0)X\|} \leq \frac{\inf_{X \in \tau_\alpha} \|X\|}{\inf_{X \in \tau_\alpha} \|DF(0)X\|} \leq \|(DF(0))^{-1}\|.$$

From (8), (12) and (13) we obtain

$$\tilde{k} \limsup_{\alpha \rightarrow 0} \frac{1 - \|(DF(0))^{-1}\| \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\|}{1 + (k + \varepsilon_\alpha) \|(DF(0))^{-1}\| \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\|} \leq \tilde{k}^* \leq$$

$$\leq \tilde{k} \limsup_{\alpha \rightarrow 0} \frac{1 + \|(DF(0))^{-1}\| \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\|}{1 - (k + \varepsilon_\alpha) \|(DF(0))^{-1}\| \sup_{X \in \tau_\alpha} \|\varepsilon(0, X)\|}.$$

Since $\lim_{\|X\| \rightarrow 0} \|\varepsilon(0, X)\| = 0$, it follows that $\tilde{k} \leq \tilde{k}^* \leq \tilde{k}$, hence $\tilde{k}^* = \tilde{k}$.

The necessity is a consequence of the sufficiency. We suppose that F maps the regular family $\{\tau_\alpha\}$ of parameter k , relatively to 0, in a regular family of parameter $\tilde{k}$ and $DF(0)$ maps $\{\tau_\alpha\}$ in a regular family of parameter $\tilde{k}$. By the sufficiency we have $\tilde{k} = \tilde{k}^*$.

Proposition 3. A regular homeomorphism $f: V \rightarrow \tilde{V}$, maps any regular hypersurface family $\{\sigma_\alpha\}$, ($0 < \alpha < 1$) of parameter k relatively to an A-point $x \in V$, into a regular family $\{\tilde{\sigma}_\alpha\}$ of parameter $\tilde{k}$, iff $T_x f$ maps the regular family $\{\tau_\alpha\}$, $\tau_\alpha = \exp_x^{-1}(\sigma_\alpha)$ of parameter k relatively to 0_x , into a regular family of parameter $\tilde{k}$.

Proof. If $x \in V$ is an A-point of f , then 0_x is an A-point of $F_x = \exp_x^{-1} \circ f \circ \exp_x$ since $T_x f = DF_x(0_x)$.

Let $\{\sigma_\alpha\}$ be a regular hypersurface family of parameter k relatively to x , then $\{\tau_\alpha\}$ is a regular family of parameter k , relatively to 0_x (Remark 4). If $\{f(\sigma_\alpha)\}$ is a regular family of parameter $\tilde{k}$, relatively to $\tilde{x} = f(x)$, it follows that $\{\exp_{\tilde{x}}^{-1}(f(\sigma_\alpha))\} = \{F_x(\tau_\alpha)\}$ is a regular family of parameter $\tilde{k}$, relatively to $0_{\tilde{x}}$. By Proposition 2, we have that $T_x f = DF_x(0_x)$ maps the regular family $\{\tau_\alpha\}$ of parameter k , into a regular family of parameter $\tilde{k}$.

Conversely, if $\{\tau_\alpha\}$ is a regular family of parameter k , relatively to $0_x \in T_x V$ and $T_x f$ maps $\{\tau_\alpha\}$ into a regular family of parameter $\tilde{k}$, then by Proposition 2, it follows that F_x maps the regular family $\{\tau_\alpha\}$ of parameter k relatively to 0_x , into a regular family of parameter $\tilde{k}$, hence f has the same property.

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ELIPSOIZI ȘI APLICAȚII REGULATE ÎN VARIETĂȚI RIEMANNIENE

(Rezumat)

Scopul acestei Note este de a dovedi că un homeomorfism între două varietăți riemanniene infinit dimensionale este regulat dacă și numai dacă aplicația sa liniară tangentă este regulată.

AN EXTENSION OF AN INEQUALITY OF G. PÓLYA

BY

HORST ALZER

In the classical book „Problems and Theorems in Analysis, I“ by G. Pólya and G. Szegő ([1], p. 57 and p. 24) one can find a proof for the following

Proposition. Let the function $f: [0, 1] \rightarrow \mathbf{R}$ be nonnegative, increasing and $f(x) \neq 0$. If u and v are nonnegative real numbers, then

$$(1) \quad 1 - \left(\frac{u-v}{u+v+1} \right)^2 \leq \frac{\left(\int_0^1 x^{u+v} f(x) dx \right)^2}{\int_0^1 x^{2u} f(x) dx \int_0^1 x^{2v} f(x) dx} \leq 1.$$

The right-hand side of (1) is a special case of the famous Cauchy-Schwarz inequality and the left-hand side is due to G. Pólya who published a proof in 1920.

The aim of this Note is to present a generalization of Pólya's inequality. We establish the following

Theorem. Let $f: [a, b] \rightarrow \mathbf{R}$ be nonnegative and increasing, and let $g: [a, b] \rightarrow \mathbf{R}$ and $h: [a, b] \rightarrow \mathbf{R}$ be nonnegative and increasing functions with a continuous first derivative. If $g(a) = h(a)$ and $g(b) = h(b)$; then

$$(2) \quad \int_a^b g'(t) f(t) dt \int_a^b h'(t) f(t) dt \leq \left(\int_a^b \sqrt{g'(t) h'(t)} f(t) dt \right)^2.$$

Proof. Integration by parts yields

$$\int_a^b [g'(t) + h'(t)] f(t) dt = [g(b) + h(b)] f(b) - [g(a) + h(a)] f(a) - \int_a^b [g(t) + h(t)] df(t).$$

Using the elementary inequality $x + y \geq 2\sqrt{xy}$ ($x \neq y$) we obtain

$$\int_a^b [g'(t) + h'(t)] f(t) dt \leq [g(b) + h(b)] f(b) - [g(a) + h(a)] f(a) - 2 \int_a^b \sqrt{g(t)h(t)} f'(t) dt.$$

Because of

$$\int_a^b \sqrt{g(t)h(t)} df(t) = f(b) \sqrt{g(b)h(b)} - f(a) \sqrt{g(a)h(a)} - \int_a^b (\sqrt{g(t)h(t)})' f(t) dt,$$

we get

$$\int_a^b [g'(t) + h'(t)] f(t) dt \leq [g(b) + h(b)] f(b) - 2 \int_a^b \sqrt{g(t)h(t)} f'(t) dt.$$

Since $g(a) = h(a)$ and $g(b) = h(b)$ the last inequality can be written as

$$\int_a^b g'(t) f(t) dt + \int_a^b h'(t) f(t) dt \leq 2 \int_a^b (\sqrt{g(t)h(t)})' f(t) dt,$$

which leads immediately to

$$\int_a^b g'(t) f(t) dt \leq \int_a^b h'(t) f(t) dt \leq \left(\int_a^b (\sqrt{g(t)h(t)})' f(t) dt \right)^2.$$

This proves the Theorem.

Remark. If we set in (2): $a=0$, $b=1$, $g(t)=t^{2u+1}$ and $h(t)=t^{2v+1}$ with $u \geq 0$ and $v \geq 0$, then we obtain the left-hand side of (1).

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O EXTENSIE A UNEI INEGALITĂȚI A LUI G. PÓLYA

(Rezumat)

Se demonstrează o teoremă prin care se obține o generalizare a inegalității (1) datorată lui G. Pólya.

ON SOME INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS

BY

JOSIP E. PEČARIĆ and SEVER S. DRAGOMIR

I. V. I. Levin and S. B. Stečkin proved the following theorem (see for example [1], pp. 414-415).

Theorem. Let f be defined on $[0, 1]$ satisfying the conditions:

(i) $f(x)$ is nondecreasing for $0 \leq x \leq \frac{1}{2}$,

and

(ii) $f(x) = f(1-x)$, $x \in [0, 1]$.

Then for any convex function Φ we have

$$\int_0^1 f(x)\Phi(x)dx \leq \int_0^1 f(x)dx \int_0^1 \Phi(x)dx.$$

In this section we shall prove the following generalization of this result.

Theorem 1. Let $\mu: [0, 1] \rightarrow \mathbb{R}$ be an increasing function such that $\mu(x) = -\mu(1-x)$, and let $f: [0, 1] \rightarrow \mathbb{R}$ be an integrable function with respect to μ such that (i) and (ii) hold. Then for any continuous convex function Φ we have

$$(1.1) \quad \int_0^1 d\mu(x) \int_0^1 f(x)\Phi(x)d\mu(x) \leq \int_0^1 f(x)d\mu(x) \int_0^1 \Phi(x)d\mu(x).$$

Proof. It is known that inequality

$$\int_0^1 \Phi(x)d\lambda(x) \geq 0,$$

holds for any continuous convex function Φ if and only if (see [1], p. 407):

$$(1.3) \quad \int_0^1 d\lambda(x) = 0,$$

$$(1.4) \quad \int_0^1 x d\lambda(x) = 0,$$

$$(1.5) \quad - \int_0^t (x-t) d\lambda(x) = \int_t^0 (x-t) d\lambda(x) \geq 0, \quad t \in [0, 1].$$

Using the substitution $\lambda(x) = \int_0^x (A(f; \mu) - f(t)) d\mu(t)$, where

$$A(f; \mu) := \frac{\int_0^1 f(t) d\mu(t)}{\int_0^1 d\mu(t)},$$

inequality (1.1) can be written in the form (1.2).

Now, we shall show that conditions (1.3), (1.4) and (1.5) hold. Obviously (1.3) is true. We also have

$$\int_0^1 x d\mu(x) = \int_0^{1/2} x d\mu(x) + \int_{1/2}^1 (1-x) d\mu(x) = \int_0^{1/2} d\mu(x) = \frac{1}{2} \int_0^1 d\mu(x),$$

because $\mu(x) = -\mu(1-x)$. So, we obtain

$$\begin{aligned} \int_0^1 x(A(f; \mu) - f(x)) d\mu(x) - \int_0^1 x f(x) d\mu(x) &= \frac{1}{2} A(f; \mu) \int_0^1 d\mu(x) - \int_0^1 x f(x) d\mu(x) = \\ &= \int_0^1 \left(\frac{1}{2} - x\right) f(x) d\mu(x) = 0, \end{aligned}$$

because $f(x) = f(1-x)$ and $\mu(x) = -\mu(1-x)$, i.e. (1.4.) is also true.

Condition (1.5) can be rewritten as

$$(1.5') \quad \int_0^t (x-t) d\lambda(x) \leq 0, \quad t \in [0, \frac{1}{2}],$$

$$(1.5'') \quad \int_t^1 (x-t) d\lambda(x) \geq 0, \quad t \in [\frac{1}{2}, 1].$$

The following identities

$$\int_0^t (x-t) d\lambda(x) = - \int_0^t (\lambda(x) - x(0)) dx, \quad \int_t^1 (x-t) d\lambda(x) = \int_t^1 (\lambda(1) - \lambda(x)) dx,$$

hold. So, if inequalities

$$(1.6) \quad \lambda(x) \geq \lambda(0), \quad x \in [0, \tfrac{1}{2}],$$

$$(1.7) \quad \lambda(x) \leq \lambda(1), \quad x \in [\tfrac{1}{2}, 1]$$

hold, then (1.5) is also valid. In our case (1.6) becomes

$$(1.8) \quad \int_0^x (A(f; \mu) - f(t)) d\mu(t) \geq 0.$$

Since

$$A(f; \mu) = \frac{2 \int_0^{1/2} f(t) d\mu(t)}{2 \int_0^{1/2} d\mu(t)} = \frac{\int_0^{1/2} f(t) d\mu(t)}{\int_0^{1/2} d\mu(t)},$$

(1.8) becomes

$$(1.9) \quad \frac{\int_0^{1/2} f(t) d\mu(t)}{\int_0^{1/2} d\mu(t)} \geq \frac{\int_0^x f(t) d\mu(t)}{\int_0^x d\mu(t)},$$

what is true, because f is a nondecreasing function on $[0, \tfrac{1}{2}]$ ((1.9) is the known inequality of P. Lóv é r a [2]). Thus (1.6) is valid. Analogously, we can prove that (1.7) is also valid.

2. Further, we shall give a generalization of the well known inequality of J e n s e n and H a d a m a r d for the continuous convex (concav) functions defined on an interval I of real numbers

$$(2.1) \quad \Phi\left(\frac{x+y}{2}\right) \leq (\geq) A(\Phi; x, y) \leq (\geq) \frac{\Phi(x) + \Phi(y)}{2},$$

where $x, y \in I (x \neq y)$ and $A(\Phi; x, y) = 1/(y-x) \int_x^y \Phi(t) dt$ ([4] p. 16, [3]).

Theorem 2. Let $\Phi: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a continuous convex (concav) function on I and $\lambda: [a, b] \rightarrow [0, 1]$ a Riemann integrable function on $[a, b]$. Then for every $x, y \in I$

$$(2.2) \quad \begin{aligned} \Phi[A(\lambda; a, b)x + (1-A(\lambda; a, b))y] &\leq (\geq) A[\Phi \circ (x\lambda + y(1-\lambda)); a, b] \leq \\ &\leq (\geq) A(\lambda; a, b)\Phi(x) + (1-A(\lambda; a, b))\Phi(y). \end{aligned}$$

P r o o f. Let x, y in I . Then for any $t \in [a, b]$ we have

$$\Phi[\lambda(t)x + (1 - \lambda(t))y] \leq (\geq) \lambda(t)\Phi(x) + (1 - \lambda(t))\Phi(y)$$

which implies

$$\int_a^b \Phi[\lambda(t)x + (1 - \lambda(t))y] dt \leq (\geq) \int_a^b \lambda(t) dt \Phi(x) + \left(b - a - \int_a^b \lambda(t) dt\right) \Phi(y),$$

from where results the second part of (2.2).

On the other hand, by J e n s e n's inequality, we deduce

$$\begin{aligned} \Phi \left[\frac{1}{b-a} \left(\int_a^b \lambda(t) dt x + \left(b - a - \int_a^b \lambda(t) dt \right) y \right) \right] &\leq (\geq) \\ &\leq (\geq) \frac{1}{(b-a)} \int_a^b \Phi[\lambda(t)x + (1 - \lambda(t))y] dt, \end{aligned}$$

and the theorem is proven.

If $[a, b] = [0, 1]$, the relation (2.2) can be rewritten as :

$$(2.3) \quad \Phi \left[\left(\int_0^1 \lambda(t) dt \right) x + \left(1 - \int_0^1 \lambda(t) dt \right) y \right] \leq (\geq) \int_0^1 \Phi[\lambda(t)x + (1 - \lambda(t))y] dt \leq (\geq)$$

$$\leq (\geq) \int_0^1 \lambda(t) dt \Phi(x) + \left[1 - \int_0^1 \lambda(t) dt \right] \Phi(y),$$

which gives for $\lambda(t) = t$, $t \in [0, 1]$

$$(2.4) \quad \Phi \left(\frac{x+y}{2} \right) \leq (\geq) \int_0^1 \Phi(tx + (1-t)y) dt \leq (\geq) \frac{\Phi(x) + \Phi(y)}{2},$$

which is equivalent with the J e n s e n - H a d a m a r d inequality.

Remark. Putting in T h e o r e m 2 $\lambda = \sin^2 : [0, \frac{\pi}{2}] \rightarrow [0, 1]$,

we have

$$(2.5) \quad \Phi \left(\frac{x+y}{2} \right) \leq (\geq) \frac{2}{\pi} \int_0^{\pi/2} \Phi(x \sin^2 t + y \cos^2 t) dt \leq (\geq) \frac{\Phi(x) + \Phi(y)}{2},$$

since a simple calculus gives $A(\sin^2; 0, \pi/2) = 1/2$.

If we consider in T h e o r e m 2, $\Phi = \ln$, we obtain the following interesting inequality

$$(2.6) \quad A(\lambda; a, b)x + (1 - A(\lambda; a, b))y \geq \exp [A(\ln(x\lambda + y(1 - \lambda)); a, b)] \geq x^{A(\lambda; a, b)} y^{(1 - A(\lambda; a, b))},$$

where x, y are two given elements in $(0, \infty)$.

Now, we shall point out another result of the above type.

Theorem 3. Let $\Phi: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a continuous convex (concave) function on I and $x, y: [a, b] \rightarrow I$ two Riemann integrable functions on $[a, b]$. Then for any $\lambda \in [0, 1]$

$$(2.7) \quad \Phi[\lambda A(x; a, b) + (1 - \lambda)A(y; a, b)] \leq (\geq) A(\Phi \circ (\lambda x + (1 - \lambda)y); a, b) \leq (\geq) \leq (\geq) \lambda A(\Phi \circ x; a, b) + (1 - \lambda)A(\Phi \circ y; a, b).$$

Proof. We start from the following inequality

$$\Phi[\lambda x(t) + (1 - \lambda)y(t)] \leq (\geq) \lambda \Phi(x(t)) + (1 - \lambda)\Phi(y(t)),$$

$t \in [a, b]$, $\lambda \in [0, 1]$, which gives by a similar argument to that in the proof of the above theorem the relation (2.7). We omit the details.

Finally, we have

Theorem 4. Let $\Phi: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a continuous convex (concave) function on I , $x, y: [a, b] \rightarrow I$ and $\lambda: [a, b] \rightarrow [0, 1]$ three Riemann integrable functions on $[a, b]$. Then

$$(2.8) \quad \Phi[A(\lambda x; a, b) + A((1 - \lambda)y; a, b)] \leq (\geq) A(\Phi \circ (\lambda x + (1 - \lambda)y); a, b) \leq (\geq) \leq (\geq) A(\lambda \Phi \circ x; a, b) + A((1 - \lambda)\Phi \circ y; a, b).$$

The proof is obvious by the following inequality

$$\Phi([\lambda(t)x(t) + (1 - \lambda(t))y(t)]) \leq (\geq) \lambda(t)\Phi(x(t)) + (1 - \lambda(t))\Phi(y(t)),$$

$t \in [a, b]$ and by Jensen's inequality. We omit the details.

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ASUPRA UNOR INEGALITĂȚI INTEGRALE PENTRU FUNCȚII CONVEXE

(Rezumat)

Se dau câteva generalizări ale inegalităților integrale Levin-Stečkin și Jensen-Hadamard pentru funcții convexe pe un interval de numere reale.

$$(1) : a, b \in L, \exp \{t(b-a)\} \leq \exp \{t(a-b)\} \quad (2.6)$$

$$\exp \{t(b-a)\} \leq \exp \{t(a-b)\}$$

where a, b are two given elements in L .

Now we shall prove our main theorem in the next section.

THEOREM 2. Let $\Phi : L \rightarrow L$ be a continuous convex function on L and $x, y \in L$. Then the following inequality holds for any $t \in [0, 1]$:

$$\Phi(ta + (1-t)b) \leq t\Phi(a) + (1-t)\Phi(b) \quad (2.7)$$

PROOF. We start from the following inequality:

$$\Phi(ta + (1-t)b) \leq t\Phi(a) + (1-t)\Phi(b) \quad (2.8)$$

where $a, b \in L$, $t \in [0, 1]$, which gives by a simple argument (cf. [1], p. 10) the desired result. We shall now prove the following:

Finally, we have

Theorem 2. Let $\Phi : L \rightarrow L$ be a continuous convex function on L , $x, y \in L$ and $t \in [0, 1]$. Then the following inequality holds:

$$\Phi(ta + (1-t)b) \leq t\Phi(a) + (1-t)\Phi(b) \quad (2.9)$$

The proof is obvious by the following inequality:

$$\Phi(ta + (1-t)b) \leq t\Phi(a) + (1-t)\Phi(b) \quad (2.10)$$

Let $a, b \in L$ and by 1. we have a inequality. We omit the details.

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ASYMPTOTIC INEQUALITIES FOR INTEGRAL INEQUALITIES

(Received)

So far there has been no systematic study of the asymptotic behavior of the integral inequalities. In this paper we shall study the asymptotic behavior of the integral inequalities.

ON A CLASS OF MULTIVALUED FUNCTIONS DEFINED ON $\mathbb{Q}_+^2$

BY

M. GONTINEAC

1. Introduction. This paper deals with a class of multivalued functions defined on the cartesian product $\mathbb{Q}_+^2 = \mathbb{Q}_+ \times \mathbb{Q}_+$, where $\mathbb{Q}_+$ is the set of positive rational numbers and valued in the set of real sequences.

On this class of functions we shall define a class of linear operators, called partially derivation operators, and we shall prove some properties of them.

By such a multivalued function, to each ordered pair (mh, nh) of positive rational numbers, corresponds a real sequence denoted $a_{mh, nh}^k$ where k are the indices of terms of the sequence, $(m, n) \in \mathbb{N}^2$, $k \in \mathbb{N}^*$, and $h = (k!)^{-1}$ is the step of the sequence.

2. Partially Derivation Operators of First Order. Definition 1. An operator denoted by $\nabla_p^m a_{mh, nh}^k$, is called a *p-mode partially derivation operator of first order*, of the sequence $a_{mh, nh}^k$, with respect to the index mh , if

$$(1) \quad \nabla_p^m a_{mh, nh}^k = \sum_{i=0}^p \alpha_i a_{(m+i)h, nh}^k,$$

where $\alpha_0, \alpha_1, \dots, \alpha_p$ are real numbers satisfying

$$(1.1) \quad \sum_{i=0}^p \alpha_i = 0,$$

$$(1.2) \quad \sum_{i=0}^p i \alpha_i = \frac{1}{h}, \quad h = \frac{1}{kl}.$$

Definition 2. An operator denoted by $\nabla_q^n a_{mh, nh}^k$, is called a *q-mode partially derivation operator of first order*, of the sequence $a_{mh, nh}^k$, with respect to the index nh , if

$$(2) \quad \nabla_q^n a_{mh, nh}^k = \sum_{i=0}^q \alpha_i a_{mh, (n+i)h}^k,$$

where $\alpha_0, \alpha_1, \dots, \alpha_q$ are real numbers satisfying conditions of the same type as (1.1) and (1.2).

Definition 3. Two operators are called to be equal if the coefficients with the same index are equal.

Remark. The sum and the product with real scalars of partially derivation operators are defined in an usual manner. Let us remark that a linear combination of partially derivation operators may be not a partially derivation operator.

Definition 4. An operator denoted by δp_m is called a *p-mode basic partially derivation operator of first order*, of the sequence $a_{mh,nh}^k$, with respect to the index m , if

$$(3) \quad \delta p_m a_{mh,nh}^k = \frac{a_{(m+p)h,nh}^k - a_{mh,nh}^k}{ph}, \quad p \in \mathbb{N}^*.$$

Definition 5. An operator denoted by δp_n is called a *p-mode basic partially derivation operator of first order*, of the sequence $a_{mh,nh}^k$, with respect to the index n , if

$$(4) \quad \delta p_n a_{mh,nh}^k = \frac{a_{mh,(n+p)h,nh}^k - a_{mh,nh}^k}{ph}, \quad p \in \mathbb{N}^*.$$

Lemma 1. A linear combination of two partially derivation operators of first order of $a_{mh,nh}^k$, and whose sum of coefficients equals 1, is a partially derivation operator of first order of $a_{mh,nh}^k$.

Proof. Let $\nabla_{p_j} a_{mh,nh}^k$, $j=1, 2$, two partially derivation operators of first order of $a_{mh,nh}^k$. We shall prove that

$\lambda \nabla_{p_1} a_{mh,nh}^k + (1-\lambda) \nabla_{p_2} a_{mh,nh}^k$, is a partially derivation operator of first order of $a_{mh,nh}^k$.

In view of the Definition 1, we have

$$\nabla_{p_j} a_{mh,nh}^k = \sum_{i=0}^{p_j} \alpha_{ij} a_{(m+i)h,nh}^k,$$

where

$$(1.3) \quad \sum_{i=0}^{p_j} \alpha_{ij} = 0, \quad j=1, 2,$$

$$(1.4) \quad \sum_{i=0}^{p_j} i \alpha_{ij} = \frac{1}{h}, \quad j=1, 2.$$

Then

$$\lambda \nabla_{p_1} a_{mh,nh}^k + (1-\lambda) \nabla_{p_2} a_{mh,nh}^k = \lambda \sum_{i=0}^{p_1} \alpha_{i1} a_{(m+i)h,nh}^k + (1-\lambda) \sum_{i=0}^{p_2} \alpha_{i2} a_{(m+i)h,nh}^k.$$

As a consequence of (1.3) and (1.4) it follows

$$\lambda \sum_{i=0}^{p_1} \alpha_{i1} + (1-\lambda) \sum_{i=0}^{p_2} \alpha_{i2} = 0,$$

respectively,

$$\lambda \sum_{i=0}^{p_1} i \alpha_{i_1} + (1-\lambda) \sum_{i=0}^{p_2} i \alpha_{i_2} = \frac{1}{h}.$$

Therefore, the coefficients of $a_{(m+i)h, nh}^k$ satisfying conditions of type (1.1) and (1.2), the operator

$$\lambda \nabla_{p_1}^m a_{mh, nh}^k + (1-\lambda) \nabla_{p_2}^m a_{mh, nh}^k,$$

is a partially derivation operator of first order of $a_{mh, nh}^k$.

Proposition 1. A linear combination of $s, s \in \mathbb{N}$, partially derivation operators of $a_{mh, nh}^k$, whose sum of coefficients equals 1, is a partially derivation operator of $a_{mh, nh}^k$.

Proof. Let $\nabla_{p_j}^m a_{mh, nh}^k, j \in \{1, 2, \dots, s\}$, be s partially derivation operators of $a_{mh, nh}^k$. Using the induction on s we shall prove that

$$\sum_{j=1}^s \lambda_j \nabla_{p_j}^m a_{mh, nh}^k \quad \text{satisfying} \quad \sum_{j=1}^s \lambda_j = 1,$$

is a partially derivation operator of $a_{mh, nh}^k$. In view of Lemma 1 the conclusion is true for $s=2$.

Suppose that

$$\sum_{j=1}^{s-1} \lambda'_j \nabla_{p_j}^m a_{mh, nh}^k \quad \text{satisfying} \quad \sum_{j=1}^{s-1} \lambda'_j = 1,$$

is a partially derivation operator of $a_{mh, nh}^k$. Then

$$\sum_{j=1}^s \lambda_j \nabla_{p_j}^m a_{mh, nh}^k = \sum_{j=1}^{s-1} \lambda'_j \nabla_{p_j}^m a_{mh, nh}^k + \lambda_s \nabla_{p_s}^m a_{mh, nh}^k = \mu \sum_{j=1}^{s-1} \lambda'_j \nabla_{p_j}^m a_{mh, nh}^k + \lambda_s \nabla_{p_s}^m a_{mh, nh}^k,$$

where, without loss of generality, for $\mu \neq 0$,

$$(5) \quad \lambda'_j = \frac{\lambda_j}{\mu}, \quad \mu = \sum_{j=1}^{s-1} \lambda_j.$$

From (5) it follows $\sum_{j=1}^{s-1} \lambda'_j = 1$ and, in view of the inductive hypothesis,

$$\sum_{j=1}^{s-1} \lambda'_j \nabla_{p_j}^m a_{mh, nh}^k,$$

is a partially derivation operator of $a_{mh, nh}^k$. Moreover, $\mu + \lambda_s = \sum_{j=1}^{s-1} \lambda_j + \lambda_s = 1$, by hypothesis, and, in view of the Lemma 1,

$$\sum_{j=1}^s \lambda_j \nabla_{p_j}^m a_{mh, nh}^k,$$

is a partially derivation operator of $a_{mh, nh}^k$.

Proposition 2. Every partially derivation operator of first order of $a_{mh,nh}^k$, is a unique linear combination, the sum of whose coefficients equals 1, and is basic partially derivation operators of first order of $a_{mh,nh}^k$.

Proof. Let

$$\nabla_p a_{mh,nh}^k = \sum_{i=0}^p \alpha_i a_{(m+i)h,nh}^k,$$

the coefficients α_i satisfying (1.1) and (1.2). From (3) deduce

$$(6) \quad a_{(m+p)h,nh}^k = a_{mh,nh}^k + p h \delta_p a_{mh,nh}^k,$$

and, as consequence

$$\nabla_p a_{mh,nh}^k = \sum_{i=0}^p \alpha_i (a_{mh,nh}^k + i h \delta_{im} a_{mh,nh}^k) = \sum_{i=0}^p \alpha_i a_{mh,nh}^k + \sum_{i=0}^p (i \alpha_i h) \delta_{im} a_{mh,nh}^k.$$

From (1.1) deduce

$$\sum_{i=0}^p \alpha_i a_{mh,nh}^k = 0$$

and then

$$\nabla_p a_{mh,nh}^k = \sum_{i=0}^p (i \alpha_i h) \delta_{im} a_{mh,nh}^k,$$

with $\sum_{i=0}^p i \alpha_i k = 1$ as consequence of (1.2).

Remark. The proofs of Lemma 1 and Proposition 1 and 2 involve the p -mode of derivation, with respect to the index mh . These proofs are similar for the q -mode of derivation, with respect to the index nh .

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O CLASĂ DE FUNCȚII MULTIVOCE PE Q_+^1

(Rezumat)

Se pune în evidență o clasă de funcții multivoce în care s-a definit o clasă de operatori pentru care se stabilesc o serie de proprietăți.

NONUNIFORM DICHOTOMY FOR LINEAR DIFFERENCE EQUATIONS

BY

PAVEL TALPALARU

1. Introduction. Difference equations occur in many branches of applied mathematics: numerical analysis, sample-data systems, control theory and optimization. In recent years a special attention has been paid to development of the dichotomy theory under its different aspects for difference equations parallel to that for differential equations.

The purpose of this Note is to give some sufficient conditions for the nonuniform dichotomy of the linear difference equations in a finite dimensional real euclidean space. It is discrete analogue of the nonuniform dichotomy of the linear differential equations introduced by P. Preda [2] and [3].

General results on the dichotomy property for difference equations have been obtained by G. Papaschinopoulos and J. Schinas [4]–[6].

2. Notations and Definitions. Denote by $N(n_0) = \{n_0, n_0+1, \dots\}$ and by $N_p = \{0, 1, \dots, p\}$, where $n_0, p \in \mathbb{N} \equiv \mathbb{N}(0)$; E^k the k -dimensional real euclidean space. We shall consider the linear difference equation having the form

$$(2.1) \quad x(n+1) = A(n)x(n),$$

where $A(n)$ is a $k \times k$ non-singular matrix for $n \in N(n_0)$ with elements $a_{ij}(n)$ real functions on $N(n_0)$. In what follows we denote by $|\cdot|$ any convenient norm either of a vector or of a matrix. Note that, if $X(n)$ is the fundamental matrix of (2.1), then it is the unique solution of the following matrix difference equation

$$(2.2) \quad X(n+1) = A(n)X(n), \quad X(n_0) = I,$$

where I is the identity matrix. It is easily to show that $X(n) = A(n-1)A(n-2)\dots A(n_0)I$, from which, since $A(n)$ is non-singular, it follows that $X^{-1}(n)$ exists for any $y \in N(n_0)$. Our definition and results will be given in terms of two functions $a, b: \mathbb{N} \rightarrow (0, \infty)$.

Definition 2.1. The equation (2.1) is said to possess a nonuniform exponential dichotomy on the set $\mathbb{N}$, if there exists a projection P_1 , that is a matrix such that $P_1^2 = P_1$, and constants $K > 0$, $0 < p < 1$, such that

$$(2.3) \quad \begin{aligned} |X(n)P_1X^{-1}(s)| &\leq K(a(s)/b(n))p^{n-s}, \quad n \geq s, \\ |X(n)P_2X^{-1}(s)| &\leq K(a(s)/b(n))p^{s-n}, \quad s \geq n, \end{aligned}$$

where $P_2 = I - P_1$, is the supplementary projection of P_1 .

In particular case when $a(n)$, $b(n)$ are constants, Definition 2.1 reduces to definition of exponential dichotomy for (2.1), [6]. If, in addition, $A(n)$ is a bounded matrix, then it reduces to Henry's definition ([1], p. 229).

3. Main Results. Our aim is to give some extensions of the Theorems 2.1 and 2.2 from [3] to the difference equation (2.1). The adaptation from the continuous to the discrete case is not direct but requires some special devices.

Theorem 3.1. Assume that there exist two constants $M > 1$ and $q \geq 1$ such that

$$(3.1) \quad |X(n)P_iX^{-1}(n)| \leq Ma(n)/b(n), \quad i=1, 2;$$

$$(3.2) \quad \left(\sum_{s=n}^{\infty} |a(s)X(s)P_1X^{-1}(n)|^q \right)^{1/q} + \left(\sum_{s=0}^{n-1} |a(s)X(s)P_2X^{-1}(n)|^q \right)^{1/q} \leq Ma(n),$$

for all $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. Then, the system (2.1) has a nonuniform exponential dichotomy.

Proof. For each $n \geq 1$, we consider the function

$$(3.3) \quad v(r) = a(r)^q |X(n)P_1X^{-1}(r)|^{-q}, \quad r \in N_n = \{0, 1, \dots, n\}.$$

Let us firstly consider $s \leq r \leq n$. Then taking into account (3.2) we have

$$\begin{aligned} v^{-1}(s) \sum_{r=s}^n v(r) &= a^{-q}(s) |X(n)P_1X^{-1}(s)|^q \sum_{r=s}^n a^q(r) |X(n)P_1X^{-1}(r)|^{-q} = \\ &= a^{-q}(s) \sum_{r=s}^n a^q(r) |X(n)P_1X^{-1}(r)X(r)P_1X^{-1}(s)| |X(n)P_1X^{-1}(r)|^{-q} \leq \\ &\leq a^{-q}(s) \sum_{r=s}^n a^q(r) |X(r)P_1X^{-1}(s)|^q \leq a^{-q}(s) \sum_{r=s}^{\infty} |a(r)X(r)P_1X^{-1}(s)|^q \leq M^q. \end{aligned}$$

If we denote now $h(s) = \sum_{r=s}^n v(r)$, we have

$$(3.4) \quad h(s) \leq M^q v(s), \quad \text{for all } s \leq n.$$

From here we obtain $h(s) \leq M^q(-\Delta h(s))$, where $\Delta v(s) = v(s+1) - v(s)$, and therefore $(\Delta h(s))/h(s) \leq -M^{-q}$, for all $n \geq s+1$ (i.e. $s \leq n-1$).

Replace now s by $s+1, s+2, \dots, n$ and get

$$h(s+1) \leq h(s) [1 - M^{-q}], \quad h(s+2) \leq h(s+1) [1 - M^{-q}], \dots, \quad h(n) \leq h(n-1) [1 - M^{-q}],$$

from where

$$(3.5) \quad h(n) \leq h(s) [1 - M^{-q}]^{n-s}, \quad 0 \leq n-s \leq n-1.$$

Then, from (3.4) it follows that

$$(3.6) \quad h(n) \leq [1 - M^{-q}]^{n-s} M^q v(s), \quad s \leq n-1.$$

Using (3.1) we obtain

$$(3.7) \quad h(n) = v(n) = a^q(n) |X(n) P_1 X^{-1}(n)|^{-q} \geq M^{-q} b^q(n).$$

From (3.6) and (3.7) we get

$$M^{-q} b^q(n) \leq [1 - M^{-q}]^{n-s} M^q v(s) = [1 - M^{-q}]^{n-s} M^q a^q(s) |X(n) P_1 X^{-1}(s)|^{-q},$$

i.e.,

$$(3.8) \quad |X(n) P_1 X^{-1}(s)| \leq K p^{n-s} a(s) / b(n), \quad s \leq n-1,$$

where $K = M^2 > 1$, $0 < p = [1 - M^{-q}]^{1/q} < 1$, and an estimation of the form (2.3) holds for $s \leq n-1$. When $s = n$, that estimation follows from (3.1).

Let now consider $r \geq n$. If we define the function $u(r) = a^q(r) |X(n) P_2 X^{-1}(r)|^{-q}$, for $r \geq n$, then, taking into account (3.2), we have

$$\begin{aligned} u^{-1}(s) \sum_{r=n}^s u(r) &\leq a^{-q}(s) \sum_{r=n}^s |X(n) P_2 X^{-1}(r)|^q X(r) P_2 X^{-1}(s) |^q u(r) = \\ &= a^{-q}(s) \sum_{r=n}^s a^q(r) |X(r) P_2 X^{-1}(s)|^q \leq a^{-q}(s) \sum_{r=0}^s a^q(r) |X(r) P_2 X^{-1}(s)|^q \leq M^q. \end{aligned}$$

If we denote now $g(s) = \sum_{r=n}^s u(r)$, we have

$$(3.9) \quad g(s) \leq M^q u(s), \quad \text{for all } s \geq n.$$

From here we obtain $g(s) \leq M^q \Delta g(s-1)$, for all $s \geq n-1$. Taking $s = n+1, n+2, \dots, s$, we get

$$(3.10) \quad g(s) \geq g(n) [1 - M^{-q}]^{s-n}, \quad s \geq n+1.$$

Then, from (3.9) and (3.10) it follows that

$$(3.11) \quad g(n) [1 - M^{-q}]^{s-n} \leq M^q u(s) = M^q a^q(s) |X(n) P_2 X^{-1}(s)|^{-q},$$

for all $s \geq n+1$. Using (3.1) we obtain

$$(3.12) \quad g(n) = u(n) = a^q(n) |X(n) P_2 X^{-1}(n)|^{-q} \geq M^{-q} b^q(n).$$

From (3.10) and (3.12) it follows that

$$M^{-q} b^q(n) \leq [1 - M^{-q}]^{s-n} M^q a^q(s) |X(n) P_2 X^{-1}(s)|^{-q}, \quad \text{for all } s \geq n+1,$$

i.e.,

$$(3.13) \quad |X(n) P_2 X^{-1}(s)| \leq K p^{s-n} a(s) / b(n), \quad s \geq n+1,$$

where K and p are defined like in (3.8). So an estimation of the form (2.3) holds for $s \geq n+1$. When $s = n$, that estimation follows from (3.1).

Theorem 3.2. Assume that there exist two constants $M > 1$ and $q \geq 1$ such that

$$(3.14) \quad |X(n) P_i X^{-1}(n)| \leq M a(n) / b(n), \quad i = 1, 2;$$

$$(3.15) \quad \left(\sum_{s=n}^{\infty} |b^{-1}(s) X(n) P_1 X^{-1}(s)|^q \right)^{1/q} + \left(\sum_{s=0}^{n-1} |b^{-1}(s) X(n) P_2 X^{-1}(s)|^q \right)^{1/q} \leq M b^{-1}(n),$$

for all $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. Then, the system (2.1) has a nonuniform exponential dichotomy.

Proof. For each $s \geq 1$, we consider the function

$$(3.16) \quad v(n) = b^{-q}(n) |X(n) P_1 X^{-1}(s)|^{-q}, \quad n \geq s.$$

Then, taking into account (3.15), we have

$$\begin{aligned} v^{-1}(n) \sum_{r=s}^n v(r) &\leq b^q(n) \sum_{r=s}^n |X(n) P_1 X^{-1}(r)|^q |X(r) P_1 X^{-1}(s)|^q v(r) = \\ &= b^q(n) \sum_{r=s}^n |X(n) P_1 X^{-1}(r)|^{q-b^{-q}(r)} \leq M^q. \end{aligned}$$

If we denote now $h(n) = \sum_{r=s}^n v(r)$, we have

$$(3.17) \quad h(n) \leq M^q v(n), \quad \text{for all } n \geq s.$$

From here we obtain $h(n) \leq M^q \Delta h(n-1)$, for all $n \geq s+1$. Taking $n = s+1, s+2, \dots, n$, we get

$$(3.18) \quad h(s) \leq h(n) [1 - M^{-q}]^{n-s},$$

and using (3.17), it follows

$$(3.19) \quad h(s) \leq M^q [1 - M^{-q}]^{n-s} v(n).$$

The hypothesis (3.14) and the definition of $h(n)$ imply

$$(3.20) \quad h(s) = v(s) = b^{-q}(s) |X(s) P_1 X^{-1}(s)|^{-q} \geq M^{-q} a^{-q}(s).$$

From (3.19) and (3.20) we get

$$M^{-q} a^{-q}(s) \leq [1 - M^{-q}]^{n-s} M^q b^{-q}(n) |X(n) P_1 X^{-1}(s)|^{-q}, \quad \text{for all } n \geq s+1,$$

i.e.,

$$(3.21) \quad |X(n) P_1 X^{-1}(s)| \leq K p^{n-s} a(s) / b(n), \quad \text{for all } n \geq s+1,$$

where $K = M^2 > M > 1$, $0 < p = (1 - M^{-q})^{1/q} < 1$. An estimation of the form (3.21) holds also for $s = n$ (i.e. $s \leq n \leq s+1$) and that follows from (3.14). Let now consider $n \leq s$. If we define the function $u(n) = b^{-q}(n) |X(n) P_2 X^{-1}(s)|^{-q}$, for $n \leq s$, then taking into account (3.15), we have

$$u^{-1}(n) \sum_{r=n}^s u(r) = b^q(n) \sum_{r=n}^s |X(n) P_2 X^{-1}(r)|^q |X(r) P_2 X^{-1}(s)|^q u(r) \leq M^q.$$

If we denote now $g(n) = \sum_{r=n}^s u(r)$, we have

$$(3.22) \quad g(n) \leq M^q u(n), \quad \text{for all } n \leq s.$$

From here we obtain $g(n) \leq M^q (-\Delta g(n))$, for all $n+1 \leq s$. Taking $n = n, n+1, \dots, s$, we get

$$(3.23) \quad g(s) \leq g(n) [1 - M^{-q}]^{s-n}, \quad n+1 \leq s.$$

Using (3.14) we obtain

$$(3.24) \quad g(s) = u(s) = b^{-q}(s) |X(s) P_2 X^{-1}(s)| \geq M^{-q} a^{-q}(s).$$

From (3.22)–(3.24) it follows that

$M^{-a}a^{-a}(s) \leq [1 - M^{-a}]^{s-n} M^a b^{-a}(n) |X(n)P_2 X^{-1}(s)|^{-a}$, for all $n+1 \leq s$,
i.e.,

$$(3.25) \quad |X(n)P_2 X^{-1}(s)| \leq K p^{s-n} a(s)/b(n), \quad n+1 \leq s,$$

where K and p are above defined. When $s=n$ ($n \leq s \leq n+1$), such an estimation follows from (3.14).

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DICHOTOMIE NEUNIFORMĂ PENTRU ECUAȚII LINIARE CU DIFERENȚE

(Rezumat)

Definind noțiunea de dichotomie exponențială neuniformă pentru ecuații liniare cu diferențe (Definiția 2.1) se dau, în continuare, condiții care asigură proprietatea de dichotomie exponențială neuniformă pentru asemenea sisteme (Teoremele 3.1 și 3.2).

From (3.22)–(3.24) it follows that

$$M^{\alpha}(s) \leq (1 - M^{\alpha})^{1/2} M^{\alpha}(s) |X(n)|^2, \text{ for all } n+1 \leq s, \quad (3.25)$$

$$|X(n)|^2 \leq M^{\alpha}(s) |X(n)|^2, \text{ for all } n+1 \leq s,$$

where A and p are above defined. When $n+1 \leq s \leq n+1$, such an estimation follows from (3.14).

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DICHOTOMIE NEUNIFORMĂ PENTRU ECUAȚII LINIARE CU DERIVATE (Romanian)

Definim noțiunea de dichotomie exponențială neuniformă pentru ecuații liniare cu derivate (Definiția 2.1) și dăm, în continuare, condiții care asigură proprietatea de dichotomie exponențială neuniformă pentru sistemele (Teorema 3.1 și 3.2).

THE DARBOUX PROBLEM ASSOCIATED WITH A LIPSCHITZIAN HYPERBOLIC MULTIVALUED EQUATION

BY

GEORGETA TEODORU

1. Introduction. In this paper we consider the Darboux problem [1], [2] associated with the multivalued hyperbolic equation $\partial^2 z / \partial x \partial y \in F(x, y, z)$, $F: D \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$, $D = [0, a] \times [0, b]$, F is Lipschitzian with respect to z , $F(x, y, z)$ is nonconvex set, for which we prove three existence theorems. The results are obtained by the successive approximations method, using two selection theorems, [3], [4]. This method was applied by A. F. Filippov [5], Henry Hermes [3] and C. J. Himmelberg and F. S. Van Vleck [4] for the equation $\dot{x} \in F(t, x)$. Our results are similar to those contained in [3], [4], [5].

2. Existence Theorems. Let $F: D \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a multivalued application satisfying the hypotheses:

(H₁) The values of F are contained in the ball of radius r centered in origin of $\mathbb{R}^n$;

(H₂) $F(x, y, z)$ is compact set for every $(x, y, z) \in D \times \mathbb{R}^n$;

(H₃) F is a continuous application;

(H₄) F is Lipschitzian with respect to z ; there exists a function $k: D \rightarrow \mathbb{R}_+$, $k \in L^1(D)$, such that

$$(1) \quad h(\bar{F}(x, y, z), F(x, y, z')) \leq k(x, y) \|z - z'\|, \quad (x, y) \in D, z, z' \in \mathbb{R}^n,$$

where h is the Hausdorff metric;

(H₅) The functions $\sigma \in AC([0, a]; \mathbb{R}^n)$, $\tau \in AC([0, b]; \mathbb{R}^n)$ satisfy the condition $\sigma(0) = \tau(0)$, $AC([\alpha_1, \alpha_2]; \mathbb{R}^n)$ being the space of absolutely continuous functions $f: [\alpha_1, \alpha_2] \rightarrow \mathbb{R}^n$, endowed with the norm

$$\|f\| = \sup_{t \in [\alpha_1, \alpha_2]} \|f(t)\| + \int_{\alpha_1}^{\alpha_2} \|f'(t)\| dt;$$

(H₆) There exists an absolutely continuous in the Carathéodory sense function ([6], § 565—§ 568), $\zeta: D \rightarrow \mathbb{R}^n$, $\zeta \in C^*(D; \mathbb{R}^n)$, such that

$$(2) \quad \sup_{(x, y) \in D} \rho \left(\frac{\partial^2 \zeta(x, y)}{\partial x \partial y}, F(x, y, \zeta(x, y)) \right) \leq M < +\infty, \text{ for some } M > 0.$$

Remark 1. Such functions ζ exist; for example, if ζ is constant, M may be taken r .

(H₇) The functions $\alpha: D \rightarrow \mathbb{R}^n$, $\alpha_0: D \rightarrow \mathbb{R}^n$ defined by

$$(3) \quad \alpha(x, y) = \sigma(x) + \tau(y) - \sigma(0), \quad (x, y) \in D,$$

$$(4) \quad \alpha_0(x, y) = \zeta(x, 0) + \zeta(0, y) - \zeta(0, 0), \quad (x, y) \in D,$$

satisfy the condition

$$(5) \quad \|\alpha(x, y) - \alpha_0(x, y)\| \leq M_1, \quad M_1 > 0.$$

Remark 2. The functions α and α_0 are absolutely continuous in the Carathéodory sense on D , [6]; $\alpha, \alpha_0 \in C^*(D; \mathbb{R}^n)$.

Definition, [1], [2]. The Darboux problem for the equation

$$(6) \quad \frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z), \quad (x, y, z) \in D \times \mathbb{R}^n,$$

consists in the determination of a solution for (6), which is a function $z: D \rightarrow \mathbb{R}^n$ absolutely continuous in the Carathéodory sense, [6], $z \in C^*(D; \mathbb{R}^n)$, which satisfies a.e. $(x, y) \in D$ the equation (6) and the conditions

$$(7) \quad \begin{cases} z(x, 0) = \sigma(x), & 0 \leq x \leq a, \\ z(0, y) = \tau(y), & 0 \leq y \leq b. \end{cases}$$

Theorem 1. *If the hypotheses (H₁)–(H₇) are satisfied, the Darboux problem (6)+(7) has a solution in D .*

Proof. We define the sequence of successive approximations $\{z_i\}$ $i \geq 0$: let

$$(8_0) \quad z_0(x, y) = \zeta(x, y), \quad (x, y) \in D.$$

According to Lemma 1.2 [3] there exists a measurable function $v_0: D \rightarrow \mathbb{R}^n$ such that

$$(9_0) \quad v_0(x, y) \in F(x, y, z_0(x, y)), \quad \text{a.e. } (x, y) \in D,$$

and

$$(10_0) \quad \rho\left(\frac{\partial^2 z_0(x, y)}{\partial x \partial y}, F(x, y, z_0(x, y))\right) = \left\| v_0(x, y) - \frac{\partial^2 z_0(x, y)}{\partial x \partial y} \right\|, \quad (x, y) \in D.$$

We define the second approximation by

$$(11_1) \quad \begin{aligned} z_1(x, y) &= \sigma(x) + \tau(y) - \sigma(0) + \int_0^x \int_0^y v_0(\xi, \eta) d\xi d\eta = \\ &= \alpha(x, y) + \int_0^x \int_0^y v_0(\xi, \eta) d\xi d\eta, \quad (x, y) \in D, \end{aligned}$$

from which it follows

$$(12_1) \quad \frac{\partial^2 z_1(x, y)}{\partial x \partial y} = v_0(x, y) \in F(x, y, z_0(x, y)), \text{ a.e. } (x, y) \in D.$$

Again applying the cited Lemma, it follows the existence of a measurable function $v_1: D \rightarrow \mathbb{R}^n$, having the properties

$$(9_1) \quad v_1(x, y) \in F(x, y, z_1(x, y)), \text{ a.e. } (x, y) \in D,$$

and

$$(10_1) \quad \rho\left(\frac{\partial^2 z_1(x, y)}{\partial x \partial y}, F(x, y, z_1(x, y))\right) = \left\| v_1(x, y) - \frac{\partial^2 z_1(x, y)}{\partial x \partial y} \right\|, \quad (x, y) \in D.$$

The third approximation is given by

$$(11_2) \quad z_2(x, y) = \alpha(x, y) + \int_0^x \int_0^y v_1(\xi, \eta) d\xi d\eta, \quad (x, y) \in D,$$

which implies

$$(12_2) \quad \frac{\partial^2 z_2(x, y)}{\partial x \partial y} = v_1(x, y) \in F(x, y, z_1(x, y)), \text{ a.e. } (x, y) \in D.$$

In this way we obtain the function sequences $\{z_i\}_{i \geq 0}$, $\{v_i\}_{i \geq 0}$, $z_i, v_i: D \rightarrow \mathbb{R}^n$, which satisfy

$$(9_i) \quad v_i(x, y) \in F(x, y, z_i(x, y)), \text{ a.e. } (x, y) \in D, \quad i=0, 1, 2, \dots$$

and

$$(10_i) \quad \rho\left(\frac{\partial^2 z_i(x, y)}{\partial x \partial y}, F(x, y, z_i(x, y))\right) = \left\| v_i(x, y) - \frac{\partial^2 z_i(x, y)}{\partial x \partial y} \right\|, \quad (x, y) \in D, \quad i=0, 1, 2, \dots,$$

where

$$(11_i) \quad z_i(x, y) = \alpha(x, y) + \int_0^x \int_0^y v_{i-1}(\xi, \eta) d\xi d\eta, \quad (x, y) \in D, \quad i=1, 2, \dots$$

From the preceding relation it follows

$$(12_i) \quad \frac{\partial^2 z_i(x, y)}{\partial x \partial y} = v_{i-1}(x, y) \in F(x, y, z_{i-1}(x, y)), \quad (x, y) \in D, \quad i=1, 2, \dots$$

The functions $\{v_i\}_{i \geq 0}$ are integrable from (12_i), (H₄) and (H₆). Moreover, from (1) we have

$$(13_i) \quad \rho\left(\frac{\partial^2 z_i(x, y)}{\partial x \partial y}, F(x, y, z_i(x, y))\right) = \rho(v_{i-1}(x, y), F(x, y, z_i(x, y))) \leq \rho(v_{i-1}(x, y), F(x, y, z_{i-1}(x, y))) + h(F(x, y, z_{i-1}(x, y))),$$

$$F(x, y, z_i(x, y)) \leq k(x, y) \|z_{i-1}(x, y) - z_i(x, y)\|, \quad (x, y) \in D, \quad i=2, 3, \dots,$$

and for $i=1$ holds (2) and (8₀).

After some standard calculation, using the inequality

$$(14) \quad \int_0^x \int_0^y k(s, t) \int_0^s \int_0^t k(s_1, t_1) \int_0^{s_1} \int_0^{t_1} k(s_2, t_2) \dots \int_0^{s_{n-1}} \int_0^{t_{n-1}} k(s_n, t_n) ds_n dt_n \dots ds_1 dt_1 ds dt \leq \\ \leq \frac{1}{(n+1)!} \left[\int_0^x \int_0^y k(u, v) du dv \right]^{n+1}, \quad (x, y) \in D,$$

we obtain the following basic estimations

$$(15_{i+1}) \quad \left\| \frac{\partial^2 z_{i+1}(x, y)}{\partial x \partial y} - \frac{\partial^2 z_i(x, y)}{\partial x \partial y} \right\| = \|v_i(x, y) - v_{i-1}(x, y)\| \leq \\ \leq k(x, y) \frac{M_1 + Mab}{(i-1)!} \left[\int_0^x \int_0^y k(\xi, \eta) d\xi d\eta \right]^{i-1}, \quad i=1, 2, \dots,$$

and

$$(16_{i+1}) \quad \|z_{i+1}(x, y) - z_i(x, y)\| \leq \frac{M_1 + Mab}{i!} \left[\int_0^x \int_0^y k(\xi, \eta) d\xi d\eta \right]^i, \quad i=0, 1, 2, \dots$$

From (15_{i+1}) we conclude that the sequence $\{v_i(x, y)\}_{i \geq 0}$ converges to $v: D \rightarrow \mathbb{R}^n$ in $L_\infty(D)$, and the sequence $\{z_i(x, y)\}_{i \geq 0}$ is uniformly convergent to $z: D \rightarrow \mathbb{R}^n$.

Letting $i \rightarrow \infty$ in (10_i), (11_i), (12_i) and using the hypotheses (H₂), (H₃) it follows that the limit z is an absolutely continuous function, $z \in C^1(D; \mathbb{R}^n)$, [6], and satisfies the Darboux problem (6)+(7). We obtain

$$(10) \quad \varphi\left(\frac{\partial^2 z(x, y)}{\partial x \partial y}, F(x, y, z(x, y))\right) = \left\| v(x, y) - \frac{\partial^2 z(x, y)}{\partial x \partial y} \right\|, \quad (x, y) \in D,$$

$$(11) \quad z(x, y) = \alpha(x, y) + \int_0^x \int_0^y v(\xi, \eta) d\xi d\eta, \quad (x, y) \in D,$$

$$(12) \quad \frac{\partial^2 z(x, y)}{\partial x \partial y} = v(x, y) \in F(x, y, z(x, y)), \quad (x, y) \in D.$$

Taking into account (11) and (12), the function given by (11) satisfies (6)+(7).

Theorem 2. We suppose that F satisfies the hypotheses (H₁)–(H₅), (H₇) and

(H₆) There exists an absolutely continuous function [6], $\zeta: D \rightarrow \mathbb{R}^n$, $\zeta \in C^1(D; \mathbb{R}^n)$ such that

$$(2') \quad \varphi\left(\frac{\partial^2 \zeta(x, y)}{\partial x \partial y}, F(x, y, \zeta(x, y))\right) < \varepsilon, \quad (x, y) \in D, \text{ for some } \varepsilon > 0.$$

Then there exists a solution of the Darboux problem (6)+(7), satisfying

$$(17) \quad ||z(x, y) - \zeta(x, y)|| \leq (M_1 + \varepsilon ab) \exp \left[\int_0^x \int_0^y k(\xi, \eta) d\xi d\eta \right], \quad (x, y) \in D.$$

The proof is similar to that one of Theorem 1; we obtain

$$(16'_{i+1}) \quad ||z_{i+1}(x, y) - z_i(x, y)|| \leq \frac{M_1 + \varepsilon ab}{i!} \left[\int_0^x \int_0^y k(\xi, \eta) d\xi d\eta \right]^i, \quad i=0, 1, 2, \dots,$$

and using the elementary inequality

$$\sum_{n=0}^j \frac{u^n}{n!} \leq e^u, \quad u \geq 0,$$

it follows for every $j=0, 1, 2, \dots$

$$(16''_j) \quad ||z_j(x, y) - \zeta(x, y)|| \leq (M_1 + \varepsilon ab) \exp \left[\int_0^x \int_0^y k(\xi, \eta) d\xi d\eta \right], \quad (x, y) \in D.$$

The conclusion follows letting $j \rightarrow \infty$ in the preceding relation; $z_j(x, y) \rightarrow z(x, y)$, $z: D \rightarrow \mathbb{R}^n$ uniformly for $j \rightarrow \infty$.

Theorem 3. Let $F: D \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a multivalued application satisfying the hypotheses:

- a) $F(x, y, z)$ is closed for every $(x, y, z) \in D \times \mathbb{R}^n$;
- b) $F(\dots, z)$ is measurable for each $z \in \mathbb{R}^n$, with respect to the Lebesgue measure on D ;
- c) $F(x, y, \cdot)$ is Lipschitzian with respect to z ; there exists the function $k: D \rightarrow \mathbb{R}_+$, $k \in L^1(D)$, such that

$$(1') \quad h_\alpha(F(x, y, z), F(x, y, z')) \leq k(x, y) ||z - z'||, \quad (x, y) \in D, \quad z, z' \in \mathbb{R}^n,$$

where h_α is the (generalized) Hausdorff pseudometric [4], and $(H_5), (H_6), (H_7)$.

Then the problem (6)+(7) has a solution in D .

The proof is similar to that one of Theorem 1, and use the Proposition 1 and the Corollary 1 [4] to ensure the existence of measurable functions $v_i: D \rightarrow \mathbb{R}^n$, $i=0, 1, 2, \dots$, with

$$(9'_i) \quad v_i(x, y) \in F(x, y, z_i(x, y)), \quad \text{a.e. } (x, y) \in D,$$

$$(10'_i) \quad \rho \left(\frac{\partial^2 z_i(x, y)}{\partial x \partial y}, F(x, y, z_i(x, y)) \right) = \left\| v_i(x, y) - \frac{\partial^2 z_i(x, y)}{\partial x \partial y} \right\|,$$

$$(x, y) \in D, \quad i=0, 1, 2, \dots,$$

$$(11'_i) \quad z_i(x, y) = \alpha(x, y) + \int_0^x \int_0^y v_{i-1}(\xi, \eta) d\xi d\eta, \quad (x, y) \in D, \quad i=1, 2, \dots,$$

$$(12'_i) \quad \frac{\partial^2 z_{i+1}(x, y)}{\partial x \partial y} = v_i(x, y) \in F(x, y, z_i(x, y)), \text{ a.e. } (x, y) \in D, i=0, 1, 2, \dots$$

The sequence $\{z_i(x, y)\} \ i \geq 0$ is uniformly convergent to $z: D \rightarrow \mathbb{R}^n$, and the sequence $\{v_i(x, y)\} \ i \geq 0$ converges in $L^\infty(D)$ to $v: D \rightarrow \mathbb{R}^n$.

Letting $i \rightarrow \infty$ in (11_i) and (12_i), we obtain

$$(11') \quad z(x, y) = \alpha(x, y) + \int_0^x \int_0^y v(\xi, \eta) d\xi d\eta, \quad (x, y) \in D,$$

$$(12') \quad \frac{\partial^2 z(x, y)}{\partial x \partial y} = v(x, y) \in F(x, y, z(x, y)), \text{ a.e. } (x, y) \in D.$$

To obtain the relation (12') we make use of the fact that the graph of F is closed, according to the hypotheses a), c). The function z given by (11') satisfies (7), is an absolutely continuous function on D , [6], $z \in C^1(D; \mathbb{R}^n)$ and from (12'), z is a solution of the Darboux problem (6)+(7).

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PROBLEMA LUI DARBOUX ASOCIATĂ UNEI ECUAȚII HIPERBOLICE MULTIVOCE LIPSCHITZIENE

(Rezumat)

Se consideră problema lui Darboux asociată ecuației hiperbolice multivoce $\partial^2 z / \partial x \partial y \in F(x, y, z)$, $F: D \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$, $D = [0, a] \times [0, b]$, F este lipschitziană în raport cu z și $F(x, y, z)$ este o mulțime neconvexă. Utilizând două teoreme de selecție date de H. Hermes [3] și C.J. Himmelberg și F.S. Van Vleck [4] se demonstrează trei teoreme de existență a soluției problemei lui Darboux prin metoda aproximațiilor succesive.

OSCILLATORY SOLUTIONS FOR EQUATIONS WITH DEVIATING ARGUMENTS

BY

RODICA LUCA

Modeling a large series of physical phenomena, the differential equations have been delight and rejoice by a particular interest of great mathematicians as Hale [1], Sficas [4], Staikos [4], Ladas [2], Philos [3].

Let us consider the first order differential equation with deviating argument

$$(\alpha) \quad x'(t) = f(t, x(t), x(t-\tau)), \quad t \geq 0,$$

where τ is a positive constant and f is a scalar function definite and continuous on a $D \subset \mathbf{R}^3$ domain.

Definition. Through a solution of the equation (α) we understand a function $x=x(t)$ of class C^1 that verifies the equation for any $t \geq 0$.

Associating to equation (α) the initial condition

$$(\beta) \quad x(t) = \varphi(t), \quad -\tau \leq t \leq 0,$$

we obtain the Cauchy problem.

If φ is a continuous function on the interval $[-\tau, 0]$, then for the problem $(\alpha) - (\beta)$ exists a function $x(t)$ that coincides with $\varphi(t)$ on the interval $[-\tau, 0]$ and is a solution for equation (α) , in the sense of the above definition. The existence of solution is indicated recursive on the intervals of the form $[0, \tau]$, $[\tau, 2\tau]$, ... with the aid of Peano's theorem.

Definition. Through oscillatory solution of differential equation (α) we understand a solution $x(t)$ for which exists a sequence (t_n) that tends to ∞ so that $x(t_n) = 0$, $n = 1, 2, \dots$

We further mention some known results that give sufficient conditions, any of them are necessary too, for a differential equation deviating argument to have all solutions as oscillatory.

Theorem 1 (Ladas [2]). All solutions of equation

$$(1) \quad x'(t) + P_0 x(t-\tau) = 0, \quad t \geq 0,$$

with P_0 and τ positive constants, are oscillatory if and only if

$$(1') \quad P_0 \tau e > 1.$$

Theorem 2 (Ladas [2]). *Let be the differential equation with deviating argument*

$$(2) \quad x'(t) + P(t) x(t - \tau) = 0, \quad t \geq 0,$$

where σ is a positive constant and $P(t) \geq 0$ is a continuous function on $[0, \infty)$. We assume

$$(2') \quad \begin{cases} \liminf_{t \rightarrow \infty} \int_{t-\tau}^t P(s) ds > \frac{1}{e}, \\ \liminf_{t \rightarrow \infty} \int_{t-\tau/2}^t P(s) ds > 0. \end{cases}$$

Then any solution of equation (2) is oscillatory.

Theorem 3 (Philos [3]). *Let be the differential equation with deviating argument*

$$(3) \quad x'(t) + p(t) \Phi[x(t - \tau)] = 0, \quad t \geq 0,$$

where τ is a positive constant, $p(t) \geq 0$ is a continuous function on $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, satisfying the condition

$$z \Phi(z) > 0, \quad \forall z \in \mathbb{R}^*.$$

If

$$(3') \quad \liminf_{t \rightarrow \infty} \int_{t-\tau}^t p(s) ds > \frac{1}{e} \max \left\{ \limsup_{z \rightarrow 0^+} \frac{z}{\Phi(z)}, \limsup_{z \rightarrow 0^-} \frac{z}{\Phi(z)} \right\},$$

then any solution of equation (3) is oscillatory.

Thus, we observe that for certain equations, such as

$$\begin{aligned} & x'(t) + ax(t - \tau)e^{t^2(t - \tau)} = 0; \quad t \geq 0, \quad \tau > \frac{1}{ea}, \quad a > 0; \\ & x'(t) + x(t - \tau) \left[\sum_{\alpha=1}^n t^\alpha + \sum_{\beta=1}^m x^{2\beta}(t - \tau) \right] = 0; \quad t \geq 0; \quad n, m \in \mathbb{N}, \\ (*) \quad & n \geq 1, \quad m \geq 1, \quad \tau > 0; \\ & x'(t) + ax(t - \tau) + \sum_{\substack{\alpha=1, n \\ \beta=0, m}} t^{2\alpha} x^{2\beta+1}(t - \tau) = 0; \quad t \geq 0, \quad \tau > \frac{1}{ea}, \quad a > 0; \quad n, m \in \mathbb{N}, \\ & n \geq 1, \quad m \geq 0, \end{aligned}$$

we can not apply Theorem 3, therefore we know nothing yet about the nature of solutions of these equations.

In what follows we give a theorem in the sense of above considerations that establishes sufficient conditions for the equation

$$(E) \quad x'(t) + \Phi[t, x(t - \tau)] = 0, \quad t \geq 0, \quad \tau > 0,$$

to have oscillatory solutions.

Theorem 4. If Φ is defined and continuous function on $[0, \infty) \times \mathbb{R}^*$ satisfying the conditions:

$$(i) \quad z\Phi(t, z) > 0, \quad \forall t \geq 0, \quad \forall z \neq 0,$$

$$(ii) \quad \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow a}} |\Phi(t, z)| > 0, \quad \forall a \neq 0,$$

and besides

$$(H) \quad \min \left\{ \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow 0+}} \frac{\Phi(t, z)}{z}, \quad \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow 0-}} \frac{\Phi(t, z)}{z} \right\} > \frac{1}{e\tau},$$

then any solution of equation (E) is oscillatory.

Proof. We prove the theorem through reductio ad absurdum. We suppose that equation (E) have a nonoscillatory solution $x(t)$. Then we can suppose that exists $t_x \geq 0$ so that $x(t) > 0$ for $t \geq t_x$. In contrary case, with change of function $w(t) = -x(t) > 0$, equation (E) become

$$w'(t) - \Phi[t, -w(t-\tau)] = 0.$$

Denoting $\Phi_1(t, w) = -\Phi(t, -w)$ we obtain equation $w'(t) + \Phi_1[t, w(t-\tau)] = 0$ that fulfils theorem conditions

$$(i) \quad z\Phi_1(t, z) = (-z) \Phi(t, -z) > 0, \quad \forall t \geq 0, \quad \forall z \neq 0$$

$$(ii) \quad \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow a}} |\Phi_1(t, z)| = \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow -a}} |\Phi(t, z)| > 0, \quad \forall a \neq 0$$

and

$$(H) \quad \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow 0+}} \frac{\Phi_1(t, z)}{z} = \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow 0-}} \frac{\Phi(t, z)}{z} > \frac{1}{e\tau}.$$

In this way, we can draw the conclusion that in this case too, the theorem is true in a hypothesis that the theorem is verified for positive solutions.

Hence $x(t) > 0$ for $t \geq t_x$. The restriction of $x(t)$ on interval $[T_0, \infty)$ with $T_0 = t_x + \tau$ is a nonoscillatory solution for equation (E), too.

Relating to the equation (E) in the Theorem 2, we have

$$P(t) = \frac{\Phi[t, x(t-\tau)]}{x(t-\tau)}, \quad t \geq T_0.$$

Since $x(t)$ is nonoscillatory solution, according with Theorem 2, it follows

$$(**) \quad \liminf_{t \rightarrow \infty} \int_{t-\tau}^t \frac{\Phi[s, x(s-\tau)]}{x(s-\tau)} ds \leq \frac{1}{e}.$$

Thus, we observe that the solution $x(t)$ of equation (E) is a decreasing function on $[T_0, \infty)$, because $x'(t) = -\Phi[t, x(t-\tau)] < 0$ owing to condition i), where $z = x(t-\tau) > 0$; consequently exists $\lim_{t \rightarrow \infty} x(t)$. Moreover, we have

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

Indeed, if we assume through reductio ad absurdum that $\lim_{t \rightarrow \infty} x(t) \neq 0$, then exists $a > 0$ to that $\lim_{t \rightarrow \infty} (t) = a$; by integrating the equation (E) from $t - \tau$ to t we obtain with the mean theorem :

$$-x(t) + x(t - \tau) = \int_{t-\tau}^t \Phi[s, x(s - \tau)] ds = \tau \Phi[\xi_t, x(\xi_t - \tau)] ; \quad t - \tau \leq \xi_t \leq t.$$

Passing at the limit for $t \rightarrow \infty$ we obtain, if we take account of condition ii)

$$0 = \tau \lim_{t \rightarrow \infty} [\Phi(\xi_t, x(\xi_t - \tau))] \geq \tau \liminf_{\substack{t \rightarrow \infty \\ \forall \epsilon > 0}} \Phi(t, y) > 0.$$

Thus, we have reached a contradiction, hence the assumption made is false ; then $\lim_{t \rightarrow \infty} x(t) = 0$.

Returning to inequality (**) and applying the mean theorem we obtain

$$\frac{1}{e} \geq \liminf_{t \rightarrow \infty} \int_{t-\tau}^t \frac{\Phi[s, x(s - \tau)]}{x(s - \tau)} ds \geq \tau \liminf_{\substack{t \rightarrow \infty \\ (t - \tau \leq \xi_t \leq t)}} \frac{\Phi[\xi_t, x(\xi_t - \tau)]}{x(\xi_t - \tau)} \geq \tau \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow 0+}} \frac{\Phi(t, z)}{z},$$

that is

$$\liminf_{\substack{t \rightarrow \infty \\ z \rightarrow 0+}} \frac{\Phi(t, z)}{z} \leq \frac{1}{e\tau},$$

an inequality that contradicts condition (H).

In consequence the equation has no nonoscillatory solution. In other words all solutions are oscillatory.

Remark. The equations (*) fulfil the conditions of Theorem 4, hence they have all oscillatory solutions.

Now, we consider the differential equation of n^{th} order ($n > 1$) with deviating argument

$$(\tilde{E}) \quad x^{(n)}(t) + (-1)^{n+1} \Phi[t, x(t - \tau)] = 0, \quad t \geq 0, \quad \tau > 0.$$

Thus, here we establish sufficient conditions for all bounded solutions of equation ($\tilde{E}$) to be oscillatory. For this we remember a Lemma from [3] and Philos's theorem corresponding to equation (3) from same paper.

Lemma (adaptation of corresponding Lemma from [4]). *Let f be a positive bounded function of class C^n on interval $[0, \infty)$ so that $(-1)^n f^n(t) \geq 0, \forall t \geq 0$ and $f^{(n)}$ is not identical zero on any interval of the form $[t_1, \infty), t_1 \geq 0$. Then $(-1)^k f^{(k)}(t) > 0, \forall t \geq 0, k = 1, 2, \dots, n-1$ and for any a and b with $0 \leq a \leq b$, we have*

$$f(a) \geq \frac{(-1)^{n-1} f^{(n-1)}(b)}{(y-1)!} (b-a)^{n-1}.$$

Theorem 5 (Philos [2]). *In conditions of Theorem 3 on the functions p and Φ , if*

$$\liminf_{t \rightarrow \infty} \int_{t-\tau}^t p(s) ds > \frac{2^n (n-1)!}{e \tau^{n-1}} \max \left\{ \limsup_{z \rightarrow 0+} \frac{z}{\Phi(z)}, \limsup_{z \rightarrow 0-} \frac{z}{\Phi(z)} \right\},$$

then all bounded solutions of equation

$$x^{(n)}(t) + (-1)^{n+1} p(t) \Phi[x(t-\tau)] = 0, \quad t \geq 0, \tau > 0,$$

are oscillatory.

Theorem 6. *If Φ is defined and continuous function on $[0, \infty) \times \mathbb{R}^*$, which fulfils conditions (i) and (ii) from Theorem 4 and besides that*

$$(\tilde{H}) \quad \min \left\{ \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow 0+}} \frac{\Phi(t, z)}{z}, \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow 0-}} \frac{\Phi(t, z)}{z} \right\} > \left(\frac{2}{\tau} \right)^n \frac{(n-1)!}{e},$$

then all bounded solutions of equation $(\tilde{E})$ are oscillatory.

Proof. We suppose by reductio ad absurdum that x is a nonoscillatory solution of equation $(\tilde{E})$, bounded on $[t_x, \infty)$, $t_x \geq 0$ and $x(t) > 0$, $\forall t \geq t_x$ (in contrary case it follows the reasoning from Theorem 4). Then, we have $(-1)^n x^{(n)} \geq 0$ on any interval $[\tau_x, \infty)$ with $\tau_x \geq t_x + \tau$ and $x^{(n)}$ is not identical zero, because Φ has not this property. It results from lemma that $(-1)^k x^{(k)}(t) > 0$, $\forall t \geq t_x + \tau$, $k=1, 2, \dots, n-1$. We observe that the restriction of function $(-1)^{n-1} x^{(n-1)}$ on interval $[T_0, \infty)$ where $T_0 = t_x + \frac{\tau}{2}$ is a solution on interval $[T_0, \infty)$ of equation (2); having in view the applicability of the second part of the Lemma

$$P(t) = \frac{\Phi[t, x(t-\tau)]}{(-1)^{n-1} x^{(n-1)}(t-\tau/2)}, \quad t \geq T_0,$$

where the delay τ from equation (2) was replaced by $\tau/2$. From

Theorem 2 we deduce

$$(4) \quad \begin{cases} \liminf_{t \rightarrow \infty} \int_{t-\tau/2}^t P(s) ds \leq \frac{1}{e}, \\ \liminf_{t \rightarrow \infty} \int_{t-\tau}^t P(s) ds \leq \frac{2}{e}. \end{cases}$$

From Lemma, where $a=t-\tau$, $b=t-\tau/2$, results

$$(***) \quad x(t-\tau) \geq \frac{(-1)^{n-1} \tau^{n-1}}{2^{n-1} (n-1)!} x^{(n-1)} \left(t - \frac{\tau}{2} \right), \quad t \geq T_0 + \frac{\tau}{2}.$$

With observations that x is a decreasing function on $[t_x + \tau, \infty)$ and that $\lim_{t \rightarrow \infty} x(t) = 0$, justified as in Theorem 4 and with inequalities (4), (***) become :

$$1 \leq \frac{2^{n-1}(n-1)!}{\tau^{n-1}} P(t) \frac{x(t-\tau)}{\Phi[t, x(t-\tau)]}, \quad t \geq T_0 + \frac{\tau}{2},$$

$$\tau \leq \frac{2^{n-1}(n-1)!}{\tau^{n-1}} \left[\sup_{0 < z \leq x(t-2\tau)} \frac{z}{\Phi(t, z)} \right] \int_{t-\tau}^t P(s) ds, \quad t \geq T_0 + \frac{3}{2} \tau,$$

$$\tau \leq \frac{2^n(y-1)!}{e \sigma^{n-1}} \limsup_{\substack{t \rightarrow \infty \\ t \rightarrow 0+}} \frac{z}{\Phi(t, z)} \Leftrightarrow \liminf_{\substack{t \rightarrow \infty \\ t \rightarrow 0+}} \frac{\Phi(t, z)}{z} \leq \left(\frac{2}{\tau} \right)^n \frac{(y-1)!}{e},$$

which contradicts condition $(\tilde{H})$. Hence the equation has the property that any its bounded solution is oscillatory. Q.E.D.

Remarks. 1. For application of Lemma has no interfere in an essential mode the taking into consideration of point $b=t-\tau/2$. Thus, we can take $b=t-\tau/m \geq 2$ and, following the same way of demonstration we obtain for the equations $(\tilde{E})$ of order $n \geq 3$ a weaker condition than $(\tilde{H})$

$$(\tilde{H}) \quad \min \left\{ \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow 0+}} \frac{\Phi(t, z)}{z}, \quad \liminf_{\substack{t \rightarrow \infty \\ z \rightarrow 0-}} \frac{\Phi(t, z)}{z} \right\} > \left(\frac{m}{m-1} \right)^{n-1} \frac{(n-1)!}{e} \frac{m}{\tau^n}.$$

2. Examples of Eq. $(\tilde{H})$ for which is applied the Theorem 6, but no more it is applied the Theorem 5, can be constructed in analogous way as equation $(*)$.

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SOLUȚII OSCILATORII PENTRU ECUAȚII CU ÎNȚÂRIERE

(Rezumat)

În articol se stabilesc condiții suficiente pentru ca ecuația (E) să aibă toate soluțiile oscilatorii. De asemenea se stabilesc condiții suficiente ca soluțiile mărginite ale ecuației (E) să fie oscilatorii.

(1) In connection with $\varphi(x)$, $\tau \in (0, \infty)$ and $\varphi(x) \in [0, \infty)$ we suppose that they are continuous, bounded and $\varphi(0) = 0$, $\varphi(1) = 1$. These conditions for the system (1)-(4) we obtain D. C. 517.944

$$u_1(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\lambda_1^2 t} \varphi(\lambda_1 x)}{\lambda_1^2} d\lambda_1 \quad (2)$$

ON THE CONTINUITY OF A MATHEMATICAL MODEL FOR AN ECOLOGICAL SYSTEM

BY

I. ONCIULESCU

1. Introduction. In this paper we study the continuity, with respect to the initial conditions, of a mathematical model, which describes the evolution of two populations P_1 and P_2 diffusing on R and starting initially from disjoint domains, namely R_- and R_+ . It is supposed that the population P_1 is more active, i.e. it has a greater birth rate than the population P_2 . Such a model could approximate the progress of an area populated with a sort of thistle on a green area.

The existence of solution of such a model where the populations P_1 and P_2 are structured on ages has been studied by A. Haimovici [1].

2. The Model. a) It is supposed that the population P_1 diffuses on R and has A_1 as diffusion coefficient and λ_1 as growth coefficient, both constant. It is supposed also that the evolution of P_1 is not influenced by P_2 . If we denote by $u_1(x, t)$ the density of the population P_1 at the moment t in the point x , then the diffusion equation of the population P_1 is

$$(1) \quad \frac{\partial u_1}{\partial t} - A_1^2 \frac{\partial^2 u_1}{\partial x^2} + \lambda_1 u_1 = 0,$$

with the initial density

$$(2) \quad u_1(x, 0) = \begin{cases} \varphi_1(x), & x \leq 0 \\ 0 & , \quad x > 0. \end{cases}$$

b) The population P_2 diffuses also on R , has the diffusion coefficient $A_2 = \text{constant}$ and a growth coefficient $\lambda_2 = \text{constant}$, but multiplied by $u_1(x, t)$ (the density of the population P_2). If $u_2(x, t)$ is the density of this second population then the diffusion equation of P_2 is

$$(3) \quad \frac{\partial u_2}{\partial t} - A_2^2 \frac{\partial^2 u_2}{\partial x^2} - \lambda_2 u_2 u_1 = 0,$$

with the initial density

$$(4) \quad u_2(x, 0) = \begin{cases} 0, & x < 0, \\ \varphi_2(x), & x \geq 0. \end{cases}$$

(H) In connection with $\varphi_1(x)$, $x \in (-\infty, 0]$ and $\varphi_2(x)$, $x \in [0, \infty)$ we suppose that they are continuous, bounded and $\varphi_i(0) = 0$, $i = 1, 2$.

In these conditions for the system (1)–(4) we obtain

$$(5) \quad u_1(x, t) = \frac{e^{-\lambda_1 t}}{2A_1 \sqrt{\pi t}} \int_{-\infty}^0 \varphi_1(\zeta) e^{-\frac{(\zeta-x)^2}{4A_1 t}} d\zeta,$$

$$(6) \quad u_2(x, t) = \frac{1}{2A_2 \sqrt{\pi t}} \int_0^{\infty} \varphi_2(\zeta) e^{-\frac{(\zeta-x)^2}{4A_2 t}} d\zeta -$$

$$-\lambda_2 \int_0^t \frac{d\tau}{2A_2 \sqrt{\pi(t-\tau)}} \left(\int_{-\infty}^{\infty} u_1(\zeta, \tau) u_2(\zeta, \tau) e^{-\frac{(\zeta-x)^2}{4A_2(t-\tau)}} d\zeta \right).$$

An existence theorem for this system follows considering the space U of the bounded continuous functions defined on

$$D = \{(x, t), \quad x \in R, \quad |x| < \infty, \quad 0 \leq t < \infty\},$$

with the metric $d(f_1, f_2) = \sup_{(x, t) \in D} e^{-at} |f_1(x, t) - f_2(x, t)|$ ($a > 0$ is a suitable chosen constant) and the operator

$$T: U \rightarrow U, \quad (Tf)(x, t) = \frac{1}{2A_1 \sqrt{\pi t}} \int_0^{\infty} \varphi_1(\zeta) e^{-\frac{(\zeta-x)^2}{4A_1 t}} d\zeta -$$

$$-\lambda_2 \int_0^t \frac{d\tau}{2A_2 \sqrt{\pi(t-\tau)}} \left(\int_{-\infty}^{\infty} u_1(\zeta, \tau) f(\zeta, \tau) e^{-\frac{(\zeta-x)^2}{4A_2(t-\tau)}} d\zeta \right).$$

We obtain that if the functions φ_1, φ_2 satisfy the conditions (H) then the system (1)–(4) has a unique classical solution on the domain D .

3. A Continuity Problem of the System with Respect to the Initial Conditions (2), (4). Let (u_1^0, u_2^0) be a solution of the system (1), (3) which satisfies the conditions:

$$(2_0) \quad u_1^0(x, 0) = \begin{cases} \varphi_1^0(x), & x \leq 0, \\ 0, & x > 0. \end{cases}$$

$$(4_0) \quad u_2^0(x, 0) = \begin{cases} 0, & x < 0, \\ \varphi_2(x), & x \geq 0. \end{cases}$$

If $(u_1(x, t), u_2(x, t))$ is a classical solution of the system (1)–(4) with φ_1, φ_2 satisfying the conditions (H) and $\varepsilon_i > 0$, $i = 1, 2$ are constants, we consider $v_i(x, t)$, $i = 1, 2$, so that

$$(7) \quad u_i(x, t) = u_i^0(x, t) + \varepsilon_i v_i(x, t), \quad i=1, 2.$$

Replacing $u_i(x, t)$, $i=1, 2$, of (7) in (1), (3), we obtain for $v_i(x, t)$, $i=1, 2$, the system

$$(8) \quad \begin{cases} \frac{\partial v_1}{\partial t} - A_1^2 \frac{\partial^2 v_1}{\partial x^2} + \lambda_1 v_1 = 0, \\ \frac{\partial v_2}{\partial t} - A_2^2 \frac{\partial^2 v_2}{\partial x^2} + \lambda_2 \varepsilon_1 v_1 v_2 + \lambda_2 \frac{\varepsilon_1}{\varepsilon_2} v_1 u_2^0 + \lambda_2 u_1^0 v_2 = 0, \end{cases}$$

with the conditions

$$(9) \quad v_i(x, 0) = \frac{u_i(x, 0) - u_i^0(x, 0)}{\varepsilon_i}, \quad i=1, 2.$$

The first equation of the system (8) leads to

$$(10) \quad v_1(x, t) = \frac{e^{-\lambda_1 t}}{2A_1 \sqrt{\pi t}} \int_{-\infty}^0 v_1(\zeta, 0) e^{-\frac{(\zeta-x)^2}{4A_1^2 t}} d\zeta.$$

Denoting

$$(11) \quad \begin{aligned} \delta_i &= \sup_{x \in R} |u_i(x, 0) - u_i^0(x, 0)|, \quad i=1, 2, \\ K_1^0 &= \sup_{x \in (-\infty, 0]} |\varphi_1^0(x)|, \quad K_2^0 = \sup_{x \in [0, \infty)} |\varphi_2^0(x)|, \end{aligned}$$

then from (10) one obtains

$$(12) \quad |v_1(x, t)| \leq \frac{\delta_1}{\varepsilon_1} e^{-\lambda_1 t}, \quad (x, t) \in D.$$

For $v_2(x, t)$ we find first, using the system (8),

$$(13) \quad \begin{aligned} v_2(x, t) &= \frac{1}{2A_2 \sqrt{\pi t}} \int_0^\infty v_2(\zeta, 0) e^{-\frac{(\zeta-x)^2}{4A_2^2 t}} d\zeta - \\ &- \frac{\lambda_2 \varepsilon_1}{\varepsilon_2} \int_0^t \frac{d\tau}{2A_2 \sqrt{\pi(t-\tau)}} \left(\int_{-\infty}^\infty v_1(\zeta, \tau) u_2^0(\zeta, \tau) e^{-\frac{(\zeta-x)^2}{4A_2^2(t-\tau)}} d\zeta \right) - \\ &- \lambda_2 \int_0^t \frac{d\tau}{2A_2 \sqrt{\pi(t-\tau)}} \left(\int_{-\infty}^\infty (\varepsilon_1 v_1(\zeta, \tau) + u_1^0(\zeta, \tau)) v_2(\zeta, \tau) e^{-\frac{(\zeta-x)^2}{4A_2^2(t-\tau)}} d\zeta \right). \end{aligned}$$

Taking into account (11), (12) and

$$(10') \quad \begin{aligned} |u_i^0(x, t)| &\leq K_i^0 e^{-\lambda_i t}, \quad |u_i^0(x, t)| \leq M_i^0, \quad i=1, 2, \\ \forall (x, t) \in D, \quad M_i^0, \quad i=1, 2, \text{ constants,} \end{aligned}$$

we get

$$(14) \quad |v_2(x, t)| \leq \frac{\delta_2}{2\varepsilon_2} + \frac{\lambda_2 \varepsilon_1}{\varepsilon_2} \int_0^t \frac{d\tau}{2A_2 \sqrt{\pi(t-\tau)}} \left(\int_{-\infty}^{\infty} \frac{\delta_1}{\varepsilon_1} M_1^0 e^{-\lambda_1 \tau - \frac{(\zeta-x)^2}{4A_2^2(t-\tau)}} d\zeta \right) + \\ + \lambda_2 \int_0^t \frac{d\tau}{2A_2 \sqrt{\pi(t-\tau)}} \left(\int_{-\infty}^{\infty} (\delta_1 + K_1^0) |v_2(\zeta, \tau)| e^{-\lambda_1 \tau - \frac{(\zeta-x)^2}{4A_2^2(t-\tau)}} d\zeta \right).$$

Because we have

$$(17) \quad \frac{\partial v_2}{\partial t} - A_2^2 \frac{\partial^2 v_2}{\partial x^2} + \lambda_2 (\varepsilon_1 v_1 + u_1^0) v_2 = -\lambda_2 \frac{\varepsilon_1}{\varepsilon_2} u_2^0 v_1,$$

we can apply the Theorem 2.5 from Chapter 1 of [4], using (10), (10') and obtain

$$(15) \quad |v_2(x, t)| \leq \left(\frac{\delta_2}{\varepsilon_2} + t \frac{\lambda_2}{\varepsilon_2} M_2^0 \right) e^{\alpha_0 t}, \quad \forall (x, t) \in D,$$

where $\alpha_0 > 0$ is a constant satisfying

$$(11) \quad \lambda_2 (\varepsilon_1 v_1(x, t) + u_1^0(x, t)) \geq -\alpha_0,$$

on the domain D . Because we have

$$|\lambda_2 \varepsilon_1 v_1(x, t) + u_1^0(x, t)| \leq \lambda_2 \delta_1 + M_1^0, \quad \forall (x, t) \in D,$$

we choose $\alpha_0 = \lambda_2 \delta_1 + M_1^0$.

Using (14) and (15) we obtain

$$(16) \quad |v_2(x, t)| \leq \frac{\delta_2}{\varepsilon_2} + \lambda_2 \frac{\delta_1}{\varepsilon_2} M_2^0 \frac{1}{\lambda_1} (1 - e^{-\lambda_1 t}) + \\ + \frac{\lambda_2 (\delta_1 + K_1^0)}{\varepsilon_2} \int_0^t e^{(\lambda_2 (\delta_1 + K_1^0) - \lambda_1) \tau} (\delta_2 + \tau \delta_1 M_2^0) d\tau.$$

If the condition

$$(17) \quad \lambda_1 > \lambda_2 K_1^0,$$

is satisfied and we choose $\delta_1 > 0$, sufficiently small, so that $\lambda_2 (\delta_1 + K_1^0) - \lambda_1 < 0$, then we obtain

$$(18) \quad |v_2(x, t)| \leq \frac{\delta_2}{\varepsilon_2} + \frac{\lambda_2 \delta_1}{\varepsilon_2} + \frac{\lambda_2 (\delta_1 + K_1^0)}{\varepsilon_2 (\lambda_1 - \lambda_2 (\delta_1 + K_1^0))} (\delta_2 + \delta_1 M_2^0).$$

If we denote

$$S_{\lambda_1, \lambda_2} = \left\{ (f_1, f_2), f_1: R \rightarrow R, f_1(x) = 0, x \geq 0, f_2: R \rightarrow R, \right.$$

$$\left. f_2(x) = 0, x \leq 0, f_1, f_2 \text{ continuous, bounded,} \right.$$

$$\left. \sup_{x \in R} |f_1(x)| < \frac{\lambda_1}{\lambda_2} \right\},$$

then we have proved the following continuity property of the system (1)–(4).

Theorem. *If (u_1^0, u_2^0) is a solution of the system (1)–(4) with $(u_1(x, 0), u_2^0(x, 0)) \in S_{\lambda_1, \lambda_2}$, then for every $\varepsilon_1, \varepsilon_2 > 0$, there exists $\delta_i(\varepsilon_1 \varepsilon_2, u_1^0, u_2^0) > 0, i=1, 2$, so that for every solution (u_1, u_2) with $(u_1(x, 0), u_2^0(x, 0)) \in S_{\lambda_1, \lambda_2}$, satisfying the conditions $|u_i(x, 0) - u_i^0(x, 0)| < \delta_i, i=1, 2$, we have $|u_i(x, t) - u_i^0(x, t)| < \varepsilon_i, i=1, 2, \forall (x, t) \in D$.*

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ASUPRA CONTINUITĂȚII UNUI MODEL MATEMATIC PENTRU UN SISTEM ECOLOGIC

(Rezumat)

Pentru sistemul (1), (3), care descrie evoluția a două populații P_1, P_2 , cu difuzie pe R , pornind inițial din două domenii disjuncte R_-, R_+ , se obține condiția (17) care asigură continuitatea soluțiilor sale în raport cu condițiile inițiale (H).

LA CIRCULATION ADMISSIBLE DANS UNE SOMME DIRECTE DE GRAPHEs

PAR

VIOREL GHIUȘ

Soit un graphe non orienté $G=(N, M)$, où $N \neq \emptyset$ est l'ensemble des sommets et $M \neq \emptyset$ est l'ensemble des arêtes. On définit les fonctions numériques non négatives $l[x, y]$ et $L[x, y]$ avec $l[x, y] \leq L[x, y]$, quelque soit $[x, y] \in M$ sur l'ensemble des arêtes.

On sait [6] que le réseau correspondant d'un graphe G est le triplet $R=(G, l, L)$. Pour une orientation certaine f de G ou R , on notera $G'=(N, M')$, respectif, $R'=(G', l, L)$ le graphe (réseau) orienté attaché.

On dit qu'une circulation dans le réseau non orienté R est une paire (f, F) , où f est une orientation de R avec la propriété que les degrés locaux en $R'g'(x) \geq 1$ et $g''(x) \geq 1$ pour tout $x \in N$, mais F une circulation dans R' , c'est à dire on satisfait les équations de conservations [1]

$$(1) \quad \Sigma(F(x, y) ; y \in N) - \Sigma(F(y, x) ; y \in N) = 0, \quad \forall x \in N, (x, y) \in M'.$$

Une circulation est admissible si elle satisfait la condition

$$(2) \quad l(x, y) \leq F(x, y) \leq L(x, y), \quad \forall (x, y) \in M'.$$

Nous étudierons le problème de la circulation admissible dans une somme directe de graphes et nous obtiendrons une condition suffisante pour existence.

Soit un graphe $G=(N, M)$ qui est une somme directe de graphes G_i

$$(3) \quad G = \Sigma(G_i ; 1 \leq i \leq n) ;$$

les sous-graphes $G_i=(N_i, M_i)$ avec $N_i \subset N$ et $M_i \subset (i=1, n)$ ont en commune les sommets $O_j(j=1, k)$, c'est à dire

$$\{O_j\} = \cap (N_i ; i \in I_j), \quad \text{où } I_j \subset \{1, 2, \dots, k\}, \quad j=1, \overline{k},$$

et n'ont pas les arrêtes communs $(M_i \cap M_j = \emptyset, \forall i \neq j \text{ avec } i=1, \overline{n} \text{ et } j=1, \overline{k})$.

Nous supposons que dans les réseaux $R_i=(N_i, M_i)$ avec $i=1, \overline{n}$ existent les circulations (f_i, F_i) admissibles ; elles induisent les orientations $f_i : M_i \rightarrow N_i$ en les graphes orientés attachés $G_i^f=(N_i, M_i^f)$, $i=1, \overline{n}$.

Nous définissons une orientation $f : M \rightarrow N$ en le graphe G ainsi

$$f(M_i) = f_i(M_i) := M_i^f, \quad \text{pour } i=1, \overline{n},$$

et s'obtient le graphe orienté attaché $G'=(N, M')$, où $M'=\cup(M'_i; 1 \leq i \leq n)$.

Nous construisons la circulation admissible (f, F) en le réseau $R'=(G', l, L)$ ainsi

$$(4) \quad F(x, y)=F_i(x, y), \quad \text{où } (x, y) \in M'_i \text{ avec } i=\overline{1, n}.$$

Les équations de conservations (1) à $\cup(R'_i; 1 \leq i \leq n)=R'$ sont évidemment satisfaites pour tout les sommets dans $N-\{O_1, O_2, \dots, O_k\}$; nous montrerons que les équations de conservation sont satisfaites et pour les sommets O_j avec $j=\overline{1, k}$.

Nous noterons avec $A'_{ij} (i=\overline{1, n} \text{ et } j=\overline{1, k})$ l'ensemble des arcs dans G'_i qui sort du sommet O_j et avec A''_{ij} l'ensemble des arcs dans G'_i qui entre en sommet O_j ; $A'_{ij} \neq \emptyset$ et $A''_{ij} \neq \emptyset$, puisque dans l'existence de la circulation (f, F_i) nous avons les degrés locaux $g'_i(O_j) \geq 1, g''_i(O_j) \geq 1$ pour tout $i=\overline{1, n}$ et $j=\overline{1, k}$.

Dans ce cas $F_i(A'_{ij})=F_i(A''_{ij}), i=\overline{1, n}$ et $j=\overline{1, k}$.

Soit $\cup(M'_{ij}; i \in I_j) = M'_{ij}$, c'est à dire l'ensemble des arcs dans G' qui sort du sommet O_j et $\cup(M''_{ij}; i \in I_j) = M''_{ij}$, c'est à dire l'ensemble des arcs dans G' qui entre en sommet O_j .

En sommant (4) sur M'_{ij} nous obtenons

$$F(M'_{ij})=F_i(M'_{ij})=F_i(M''_{ij})=F(M''_{ij}) \quad \text{pour } i=\overline{1, n} \text{ et } j=\overline{1, k}.$$

Par conséquent

$$F(M'_j)=F(\cup(M'_{ij}; i \in I_j))=\sum(F(M'_{ij});$$

$$i \in I_j)=\sum(F(M''_{ij}; i \in I_j))=F(\cup(M''_{ij}; i \in I_j))=F(M''_j) \quad \text{pour } j=\overline{1, k}.$$

Les contraintes des admissibles (2) sont évidemment satisfaites. Nous pouvons donc énoncer le résultat suivant.

Théorème. *S'il existe en chaque sous-réseau $R_i (i=\overline{1, n})$ par une circulation admissible, il existe une circulation admissible dans le réseau R (somme directe des sous-réseaux R_i qui n'ont pas les arêtes communs).*

Remarque 1. Ce théorème permet l'étude du problème de la circulation admissible avec une structure compliquée.

En effet, si un graphe non orienté G satisfait (3) et les sous-graphes $G_i (i=\overline{1, n})$ sont : circuits, graphes tracés arbitrairement, graphes éventails, graphes éventails fermés, écheveaux en utilisant [2] - [5] nous pouvons obtenir une circulation admissible dans G .

Exemples. Le graphe complet $G=(N, M)$ avec cinq sommets, où $N=\{x_i; i=\overline{1, 5}\}$ peut être écrit come somme directe de deux circuits disjoints-arêtes :

$$G=C_1+C_2$$

avec

$$C_1=\cup((x_i, x_{i+1}); i=\overline{1, 4}) \cup (x_5, x_1) \text{ et } C_2=\cup((x_i, x_{i+2}); i=\overline{1, 3}) \cup (x_4, x_1) \cup (x_5, x_2).$$

Si dans chaque circuit est satisfait le Théorème 1 [2], en vertu du théorème démontré, il existe une circulation admissible dans le réseau $R=(G, l, L)$.

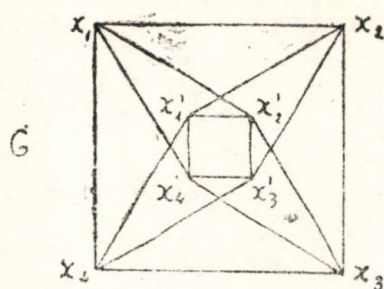


Fig. 1

Un autre exemple intéressant est le graphe G attaché au groupe des quaternions (Fig. 1).

Il a une décomposition en somme directe de circuits

$$G = \Sigma(C_i; 1 \leq i \leq 4), \text{ où}$$

$$C_i = (x_i, x_{i+1}, x'_i, x'_{i+1}, x_i) \text{ pour } i = \overline{1, 3} \text{ et}$$

$$C_4 = (x_4, x_1, x'_4, x'_1, x_4).$$

S'il existe une circulation admissible dans chaque circuit $C_i (i = \overline{1, 4})$, il existe une circulation admissible en G .

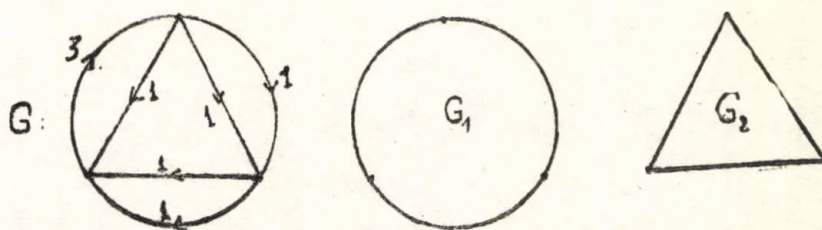


Fig. 2

Remarque 2. En général, le théorème réciproque n'est pas vrai. Ainsi, le graphe G (Fig. 2) peut être écrit $G = G_1 + G_2$.

En G existe une circulation qui n'induit pas les circulations sur G_1 et G_2 .

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CIRCULAȚIE ADMISIBILĂ ÎNTR-O SUMĂ DIRECTĂ DE GRAFURI

(Rezumat)

Se studiază problema circulației admisibile într-o sumă directă de grafuri, disjuncte muchial. Se obține o condiție suficientă de existență pentru această categorie de grafuri.

En outre, l'existence d'un tel $\mathcal{C}$ est intéressante car elle implique que le groupe des automorphismes de $\mathcal{C}$ est non trivial.

Il a été démontré en 1975 que si $\mathcal{C}$ est un $\mathcal{C}$ de type fini, alors $\mathcal{C}$ est isomorphe à $\mathcal{C}_1$ ou à $\mathcal{C}_2$.

On a également démontré que si $\mathcal{C}$ est un $\mathcal{C}$ de type fini, alors $\mathcal{C}$ est isomorphe à $\mathcal{C}_1$ ou à $\mathcal{C}_2$.

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Fig. 1

Remarque 2. En général, le $\mathcal{C}$ d'un $\mathcal{C}$ n'est pas unique. Par exemple, si $\mathcal{C}$ est un $\mathcal{C}$ de type fini, alors $\mathcal{C}$ est isomorphe à $\mathcal{C}_1$ ou à $\mathcal{C}_2$.

La $\mathcal{C}$ existe une circulation qui n'est pas la circulation qui est $\mathcal{C}$.

On a également démontré que si $\mathcal{C}$ est un $\mathcal{C}$ de type fini, alors $\mathcal{C}$ est isomorphe à $\mathcal{C}_1$ ou à $\mathcal{C}_2$.

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ON THE UNIQUENESS OF THE SOLUTION OF SOME BOUNDARY VALUE PROBLEMS OF THE LINEAR MICROPOLAR THERMOELASTICITY

BY

I. CRĂCIUN

1. Basic Equations. The basic equations of the linear micropolar thermoelasticity theory of an isotropic and homogeneous solid occupying the finite region D^+ of the three-euclidian space E_3 , bounded by the closed surface S , in the absence of body forces, couple body forces and heat sources and in the stationary vibrations case, are [1], [5]:

i) *The dynamic equations*

$$(1.1) \quad \begin{cases} t_{ji,j} + \rho \omega^2 n_i = 0, \\ m_{ji,j} + \varepsilon_{imn} t_{mn} + J \omega^2 \varphi_i = 0, \\ \Delta \cdot q = T_0 i \omega \tilde{\eta}; \end{cases}$$

ii) *The constitutive equations*

$$(1.2) \quad \begin{cases} t_{ij} = (\mu + \alpha) \gamma_{ij} + (\mu - \alpha) \gamma_{ji} + (\lambda \gamma_{kk} - \nu \theta) \delta_{ij}, \\ m_{ij} = (\gamma + \varepsilon) \kappa_{ij} + (\gamma - \varepsilon) \kappa_{ji} + \beta \kappa_{kk} \delta_{ij}, \\ q_i = k \theta_{,i} \\ \tilde{\eta} = \nu \gamma_{kk} + \frac{\lambda \varepsilon}{T_0} \theta; \end{cases}$$

iii) *The geometrical relations*

$$(1.3) \quad \begin{cases} \gamma_{ij} = u_{j,i} + \varepsilon_{jik} \varphi_k, \\ \kappa_{ij} = \varphi_{j,i}; \end{cases}$$

iv) *The Cauchy's equations*

$$(1.4) \quad \begin{cases} \mathbf{t} = t_{ji} n_j \mathbf{e}_i = t_i \mathbf{e}_i = \mathbf{e} t, & \mathbf{t} = (t_1, t_2, t_3)^T, \\ \mathbf{m} = m_{ji} y_j \mathbf{e}_i = m_i \mathbf{e}_i = \mathbf{e} m, & \mathbf{m} = (m_1, m_2, m_3)^T, \\ \mathbf{h} = q_i n_i = \mathbf{q} n, & \mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3), \end{cases}$$

where the symbols have the same meaning as in [1].

Introducing t_{ji} , m_{ji} , $\tilde{\gamma}_i$ and q_i into (1.1) and using (1.3) and the notations $\alpha = k/c_\varepsilon$, $\gamma = T_0 \nu/k$, we obtain the differential system of the micropolar thermoelastic stationary vibrations of an isotropic and homogeneous solid

$$(1.5) \quad \begin{cases} [(\mu + \alpha)\nabla^2 + \rho\omega^2]\mathbf{u} + (\lambda + \mu - \alpha)\nabla(\nabla \cdot \mathbf{u}) + 2\alpha\nabla \times \varphi - \nu\nabla\theta = \mathbf{0}, \\ 2\alpha\nabla \times \mathbf{u} + [(\gamma + \varepsilon)\nabla^2 + J\omega^2 - 4\alpha]\varphi + (\beta + \gamma - \varepsilon)\nabla(\nabla \cdot \varphi) = \mathbf{0}, \\ i\omega\gamma\nabla \cdot \mathbf{u} + \left(\nabla^2 + \frac{i\omega}{\alpha}\right)\theta = 0. \end{cases}$$

From (1.2)–(1.4) we obtain

$$(1.6) \quad \begin{cases} \mathbf{t} = \lambda \mathbf{n} \nabla \cdot \mathbf{u} + 2\mu \frac{\partial \mathbf{u}}{\partial n} + (\mu - \alpha)\mathbf{n} \times (\nabla \times \mathbf{u}) + 2\alpha \mathbf{n} \times \varphi - \nu \theta \mathbf{n}, \\ \mathbf{m} = \beta \mathbf{n} \nabla \cdot \varphi + 2\gamma \frac{\partial \varphi}{\partial n} + (\gamma - \varepsilon)\mathbf{n} \times (\nabla \times \varphi), \\ h = k \frac{\partial \theta}{\partial n}. \end{cases}$$

Let us notice that (1.5) and (1.6) can be written in the forms

$$(1.7) \quad B\left(\frac{\partial}{\partial x}, \omega\right)V(x) = 0, \quad x \in D^+,$$

$$(1.8) \quad R\left(\frac{\partial}{\partial x}, n\right)V(x) = (t_1, t_2, t_3, m_1, m_2, m_3, hk)^T,$$

where $B\left(\frac{\partial}{\partial x}, \omega\right)$ is the matrix of the differential operator of the stationary vibrations from the linear micropolar thermoelasticity of an isotropic, homogeneous and centrosymmetric solid [1], $V(x)$ is a one column matrix of the coordinates of a vector from $\mathbb{R}^7$, having the first three coordinates equal to u_1, u_2, u_3 , the following three equal to $\varphi_1, \varphi_2, \varphi_3$ and the seventh one being θ , and $R\left(\frac{\partial}{\partial x}, n\right)$

is the matrix of a boundary value differential operator of the operator $B\left(\frac{\partial}{\partial x}, \omega\right)$.

We also have

$$(1.9) \quad \begin{cases} t(x) = T\left(\frac{\partial}{\partial x}, n\right)U(x) - \nu\theta(x)\mathbf{n}, \quad n = (n_1, n_2, n_3), \quad U = (u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3)^T, \\ m(x) = M\left(\frac{\partial}{\partial x}, n\right)\varphi(x), \quad \varphi = (\varphi_1, \varphi_2, \varphi_3)^T, \end{cases}$$

where the elements of the 3×6 order matrix $T\left(\frac{\partial}{\partial x}, n\right)$ and the 3×3 order matrix $M\left(\frac{\partial}{\partial x}, n\right)$ can be easily read from (1.6).

2. Boundary Value Problems. In a previous paper [2] we defined thirty two interior boundary value problems of the linear micropolar thermoelasticity of an isotropic, homogeneous and centrosymmetric solid occupying the domain D^+ in the case of stationary vibrations. These problems have been denoted by the symbols $(p . q . r)^+$, where $p, q \in \{I, II, III, IV\}$ and $r \in \{I, II\}$.

In this Note we deal only of the problems $(p . q . r)^+$, where $p, q, r \in \{I, II\}$.

Solving such a problem is equivalent with finding a regular solution [5] $V \in C^2(D^+) \cap C^1(\bar{D}^+)$ of system (1.7) satisfying one of the following boundary condition (we make the convention to denote the boundary condition by the same symbol as the considered problem)

$$(I.I.I)^+ \quad \lim_{D^+ \ni x \rightarrow z \in S} V(x) = [V(z)]^+ = a(z),$$

$$(I.I.II)^+ \quad \left[\left[U^T(z), \frac{\partial \theta(z)}{\partial n} \right] \right]^+ = b(z),$$

$$(I.II.I)^+ \quad [(u^T(z), m^T(z), \theta(z))]^+ = c(z),$$

$$(I.II.II)^+ \quad \left[\left[u^T(z), m^T(z), \frac{\partial \theta(z)}{\partial n} \right] \right]^+ = d(z),$$

$$(II.I.I)^+ \quad [(t^T(z), \varphi^T(z), \theta(z))]^+ = e(z),$$

$$(II.I.II)^+ \quad \left[\left[t^T(z), \varphi^T(z), \frac{\partial \theta(z)}{\partial n} \right] \right]^+ = f(z),$$

$$(II.II.I)^+ \quad [(l^T(z), m^T(z), \theta(z))]^+ = g(z),$$

$$(II.II.II)^+ \quad \left[R \left(\frac{\partial}{\partial x}, n \right) V(z) \right]^+ = \bar{p}(z),$$

where $a, b, \dots, \bar{p}$ are given functions, defined on S with values in $\mathbb{R}^7$. When one of the functions $a, b, \dots, \bar{p}$ is zero, then the corresponding problem will be denoted $(p . q . r)_0^+$.

3. An Identity. In our paper [2] we proved the identity

$$(3.1) \quad \begin{aligned} & \nu \int_{D^+} (\nabla \theta(x))^2 dx + \omega \eta \mathcal{I}m \int_S \bar{u}^T(z) \left(T \left(\frac{\partial}{\partial z}, n \right) U(z) - \nu \theta(z) n(z) \right) + \\ & + \bar{\varphi}^T(z) M \left(\frac{\partial}{\partial z}, n \right) \varphi(z) \Big]^+ dS_z - \nu \mathcal{R}e \int_S \left[\bar{\theta}(z) \frac{\partial \theta(z)}{\partial n} \right]^+ dS_z = 0, \end{aligned}$$

where $\mathcal{R}e$ and $\mathcal{I}m$ are, respectively, the real and imaginary parts of the specified complex functions while a superposed bar suggest the conjugate of the complex function under bar.

4. About Uniqueness. Let us suppose that any one of the problems $(p . q . I)^+$ defined in the above has two different solutions V_1 and V_2 . Then, the

difference of these solutions $V = V_1 - V_2$ is a solution of the problem $(p, q, I)_0^+$. Writing now (3.1) for this difference, we obtain

$$(4.1) \quad \int_{D^+} (\nabla \theta(x))^2 dx = 0,$$

from which we deduce that θ is a constant in D^+ . Because $\theta \in C^2(D^+) \cap C^1(\bar{D}^+)$ and $[\theta(z)]^+ = 0$, in the case of the $(p, q, I)^+$ problems, we notice that

$$(4.2) \quad \theta(x) = 0, \quad x \in \bar{D}^+.$$

On the other hand any vector on $\mathbb{R}^3$ with values in $\mathbb{R}^3$ can be decomposed in a solenoidal part and in a potential one, so that we shall have

$$(4.3) \quad V(x) = (u^T(x), \varphi^T(x), 0)^T = (u_*^T(x), \varphi_*^T(x), 0)^T + (u_*^2(x), \varphi_*^2(x), 0)^T,$$

where

$$(4.4) \quad u_*^1 = e u_*, \quad \varphi_*^1 = e \varphi_*, \quad u_*^2 = e u_*, \quad \varphi_*^2 = e \varphi_*,$$

are the solenoidal part and the potential one of the vectors $u = e u$, $\varphi = e \varphi$, respectively. Therefore

$$(4.5) \quad \nabla \cdot u_*^1 = 0, \quad \nabla \cdot \varphi_*^1 = 0, \quad \nabla \times u_*^2 = 0, \quad \nabla \times \varphi_*^2 = 0.$$

From system (1.5) and (4.4), (4.5), we obtain [2]

$$(4.6) \quad \begin{cases} u_*^1 = \frac{1}{\sigma_2^2} \nabla \times (\nabla \times u_*^1) - \frac{s}{\sigma_2^2} \nabla \times \varphi_*, \\ \varphi_*^1 = \frac{1}{\hat{\sigma}_4^2} \nabla \times (\nabla \times \varphi_*) - \frac{p}{\hat{\sigma}_4} \nabla \times u_*, \end{cases}$$

$$(4.7) \quad u_*^2 = -\frac{1}{\sigma_1^2} \nabla (\nabla \cdot u_*^2), \quad \varphi_*^2 = -\frac{1}{\hat{\sigma}_3^2} \nabla (\nabla \cdot \varphi_*^2).$$

But, taking into account (4.2) and (1.5)₃, we get

$$(4.8) \quad u_*(x) = 0, \quad x \in D^+.$$

In this way, the solenoidal part $u = u^*$ of the displacement vector and the rotation vector $\varphi = \varphi_*^1 + \varphi^*$ satisfy the system

$$(4.9) \quad \begin{cases} [(\mu + \alpha) \nabla^2 + \rho \omega^2] u + 2\alpha \nabla \times \varphi = 0, \\ [(\gamma + \varepsilon) \nabla^2 + J \omega^2 - 4\alpha] \varphi + (\beta + \gamma - \varepsilon) \nabla (\nabla \cdot \varphi) + 2\alpha \nabla \times u = 0, \\ \nabla \cdot u = 0, \end{cases}$$

with one of the boundary conditions

$$(4.10) \quad [U(z)]^+ = 0,$$

$$(4.11) \quad \left[\left(T \left(\frac{\partial}{\partial z}, n \right) U(z) \right)^T, \left(M \left(\frac{\partial}{\partial z}, n \right) \varphi(z) \right)^T \right]^+ = 0,$$

$$(4.12) \quad \left[\left(T \left(\frac{\partial}{\partial z}, n \right) U(z) \right)^T, \varphi^T(z) \right]^+ = 0,$$

$$(4.13) \quad \left[u^T(z), \left(M \left(\frac{\partial}{\partial x}, n \right) \varphi(z) \right)^T \right]^+ = 0.$$

Therefore, the boundary value problems $(I.I.I)_0^+$, $(II.II.I)_0^+$ $(II.I.I)_0^+$, $(I.II.I)_0^+$ were reduced, respectively, to the problems $\{(4.9), (4.10)\}$, $\{(4.9), (4.11)\}$, $\{(4.9), (4.12)\}$ and $\{(4.9), (4.13)\}$.

From the results deduced in the above we deduce that the problem $(p.q.I)^+$ has unique solution if the corresponding boundary value problem $\{(4.9), (4.10)\}, \dots, \{(4.9), (4.13)\}$ has only the trivial solution.

Considering now the problems $(p.q.II)^+$ and supposing that such a problem has two different solutions $V_1(x)$ and $V_2(x)$, we derive from (3.1) that the seventh coordinate of the difference $V(x) = V_1(x) - V_2(x)$ a constant c on D^+ . By using this result in the last equation (1.5), we obtain

$$(4.14) \quad \nabla \cdot \mathbf{u}_* = -\frac{1}{x\eta} c.$$

From (4.7)₁ and (4.14), we get

$$(4.15) \quad \mathbf{u} = 0, \text{ in } D^+.$$

Returning now in (1.5)₃ we find that $\theta(x) = 0$ in D^+ .

Therefore, in the case of the problems $((p.q.II)_0^+)$ we also have $\theta = 0$, $\mathbf{u} = \mathbf{u}^*$, $\varphi = \varphi_1^* + \varphi_2^*$ and for the determination of the $\mathbf{u}$ and φ vectors we must to solve again the system (4.9) with one of the boundary conditions (4.10) — (4.13). So, the problem $(p.q.II)_0^+$ has a unique solution if the corresponding problem $\{(4.9), (4.10)\}, \dots, \{(4.9), (4.13)\}$ has only the trivial solution.

5. The Spectrum of the Proper Frequencies. The boundary value problems $\{(4.9), (4.10)\}, \dots, \{(4.9), (4.13)\}$ are particular forms of the problems $(I)_0^+$, $(II)_0^+$, $(III)_0^+$ and $(IV)_0^+$, respectively, studied by O.I. Napetvaridze in [3]. From [3] and also from the results obtained in [5] and [6], we deduce that there exists values of the frequency ω so that the problems $\{(4.9), (4.10)\}, \dots, \{(4.9), (4.13)\}$ have nontrivial solutions. These values of ω will be referred as proper frequencies of the problems $(p.q.r)^+$, $p, q, r \in \{I, II\}$. The set of the proper frequencies of the problem $(p.q.r)^+$ will be called the spectrum of the corresponding problem. The spectrum of any problem $(p.q.r)^+$ contains only positive numbers and also it is a subset of the spectrum of one of the problems $(I)_0^+$, $(II)_0^+$, $(III)_0^+$, $(IV)_0^+$.

From the results derived in this Note, we conclude that if ω is not a proper frequency of a boundary value problem here, then the corresponding problem has unique solution. Otherwise, the solution of the boundary value problem $(p \cdot q \cdot r)^+$ is not unique. But, any two solutions of the problem $(p \cdot q \cdot r)^+$, corresponding to a proper frequency ω , differ only by the displacement vector and the rotation vector, the increment of the temperature θ being the same. Furthermore, the difference between the displacement vectors of two solutions of the problem $(p \cdot q \cdot r)^+$, corresponding to a proper frequency, is a solenoidal vector.

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ASUPRA UNICITĂȚII SOLUȚIILOR UNOR PROBLEME LA LIMITĂ ÎN TERMOELASTICITATEA MICROPOLARĂ LINIARĂ

(Rezumat)

Se consideră unele probleme cu valori la limită, $(p \cdot q \cdot r)^+$, ale teoriei liniare a termoelasticității micropolare în cazul vibrațiilor staționare. Se introduce noțiunea de frecvență proprie a problemei $(p \cdot q \cdot r)^+$ și se arată că dacă ω nu este o astfel de frecvență, atunci soluția problemei $(p \cdot q \cdot r)^+$ este unică. Dacă ω este o frecvență proprie a problemei $(p \cdot q \cdot r)^+$, atunci pot exista mai multe soluții, oricare două dintre ele diferă doar prin vectorul de deplasare și prin cel al rotațiilor, în plus, diferența dintre vectorii de deplasare este un vector solenoidal.

SYMMETRIES IN THE LINEAR THERMOELASTICITY OF THE DIRECTED RODS

BY

GH. CHIORESCU

1. Introduction. In the paper [1] the basic equations for the linear thermoelastic theory have been deduced. These equations are:

i) *The deformation-displacement equations*

$$(1.1) \begin{cases} \gamma_{\alpha\beta} = D_{(\alpha)} \delta_{\beta\alpha} + D_{(\beta)} \delta_{\alpha\beta}, & \gamma_{\alpha 3} = \delta_{\alpha 3} + D_{(\alpha)} u'_{\alpha}, & \gamma_{33} = 2\delta_{33} = 2u'_3 - (A'_{33}/A_{33})u_3, \\ k_{\alpha\beta} = D_{(\beta)} \delta_{\alpha\beta} + D_{(\alpha)} \delta_{\beta\alpha}, & k_{\alpha 3} = \delta'_{\alpha 3} + D_{(\alpha)} u'_{\alpha} - (A'_{33}/2A_{33})\delta_{\alpha 3}. \end{cases}$$

ii) *The motion equations*

$$(1.2) \begin{cases} n'^{\alpha} + \lambda f^{\alpha} = \lambda \ddot{u}^{\alpha}, & n'^3 + (A'_{33}/2A_{33})n^3 + \lambda f^3 = \lambda \ddot{u}^3, \\ p^{\alpha\beta} + \lambda l^{\alpha\beta} = \pi^{\alpha\beta} + J^{\alpha}_{\gamma} \ddot{\delta}^{\gamma\beta}, & p^{\alpha 3} + (A'_{33}/2A_{33})p^{\alpha 3} + \lambda l^{\alpha 3} = \pi^{\alpha 3} + J^{\alpha}_{\gamma} \ddot{\delta}^{\gamma 3}, \\ -h' + \lambda r = \lambda T \dot{\gamma}. \end{cases}$$

iii) *The constitutive equations*

$$(1.3) \begin{cases} n^{\alpha} = D_{(\alpha)} (2A_{\alpha\beta\gamma} \gamma_{\beta\gamma} + A_{\alpha 333} \gamma_{33} + B_{\alpha\beta\gamma} k_{\beta\gamma} - E_{\alpha 3} \theta) + \\ + D'_{(\alpha)} (B_{\lambda i \alpha 3} \gamma_{\lambda i} + B_{\alpha 333} \gamma_{33} + 2C_{\alpha\beta\gamma} k_{\beta\gamma} - F_{\alpha 3} \theta), \\ n^3 = 2(A_{\alpha i 33} \gamma_{\alpha i} + 2A_{3333} \gamma_{33} + B_{\alpha 3 i 33} k_{\alpha i} - E_{33} \theta), \\ p^{\alpha\beta} = D_{(\beta)} (B_{\lambda i \alpha \beta} \gamma_{\lambda i} + B_{\alpha\beta 33} \gamma_{33} + 2C_{\alpha\beta\gamma} k_{\gamma j} - F_{\alpha\beta} \theta), \\ p^{\alpha 3} = B_{\lambda i \alpha 3} \gamma_{\lambda i} + B_{\alpha 333} \gamma_{33} + 2C_{\alpha\beta\gamma} k_{\beta\gamma} - F_{\alpha 3} \theta, \\ \pi^{\alpha\beta} = 2D_{(\beta)} (2A_{\alpha\beta\lambda\gamma} \gamma_{\lambda\gamma} + A_{\alpha\beta 33} \gamma_{33} + B_{\alpha\beta\gamma j} k_{\gamma j} - E_{\alpha\beta} \theta) + \\ + D'_{(\beta)} (B_{\lambda i \alpha \beta} \gamma_{\lambda i} + B_{\alpha\beta 33} \gamma_{33} + 2C_{\alpha\beta\gamma} k_{\gamma j} - F_{\alpha\beta} \theta), \\ \pi^{\alpha 3} = 2A_{\lambda\beta\gamma} \gamma_{\beta\gamma} + A_{\alpha 333} \gamma_{33} + B_{\alpha\beta\gamma} k_{\beta\gamma} - F_{\alpha 3} \theta, \\ \gamma_i = E_{\alpha i} \gamma_{\alpha i} + E_{33} \gamma_{33} + F_{\alpha i} k_{\alpha i} + G \theta, \end{cases} \quad (1.3)$$

and

$$(1.4) \quad h = -k\theta, \quad k > 0,$$

where

i) $A_{ij}, B_{ij}, C_{ij}, E_{ij}, F_{ij}, G$ are the constitutive coefficients which satisfy the conditions

$$(1.5) \quad \begin{cases} A_{\alpha\beta\gamma\delta} = A_{\beta\gamma\alpha\delta}; & A_{\alpha\beta\gamma\delta} = A_{\beta\gamma\delta\alpha}; & A_{\alpha\beta\gamma\delta} = A_{\beta\gamma\delta\alpha}; \\ B_{\alpha\beta\gamma\delta} = B_{\beta\gamma\alpha\delta}; & B_{\alpha\beta\gamma\delta} = B_{\beta\gamma\delta\alpha}; & C_{\alpha\beta\gamma\delta} = C_{\beta\gamma\alpha\delta}; & E_{\alpha\beta} = E_{\beta\alpha}; \end{cases}$$

ii) $\mathbf{u}$, δ_i and θ are the displacement vectors and the temperature function; γ_{ij} and k_{ij} are the deformation measures, $D_{(\alpha)}$ and T are the director vectors and the temperature function in the reference configuration; $\mathbf{A}_i$ are the base vector assumed normal to each other and

$$\mathbf{D}_\alpha = D_{(\alpha)} \mathbf{A}_\alpha; \quad \mathbf{D}_\alpha \cdot \mathbf{A}_\alpha = 0, \quad \alpha = 1, 2 \text{ (no sum on } \alpha);$$

$\mathbf{n}$, $\mathbf{p}^\alpha$, π^α , η , h are the contact force, the director contact force, the intrinsic couple vectors, the entropy and the heat flux; $\mathbf{f}$, $\mathbf{f}^\alpha$, $\mathbf{r}$ are the body force, the body director force and the heat supply; $J^{\alpha\beta}$ are the coefficients of the inertia tensor; λ is the mass density in the reference configuration.

In the following we deduce the basic equations of the theory when some material and geometrical symmetries are imposed.

2. Material and Geometrical Symmetries. We assume the rod to be material and geometrical symmetric with respect to the two orthogonal directors in planes perpendicular to the line C in the reference configuration. It follows that ψ is form-invariant under a change of the sign of the vectors; $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{D}_1$, $\mathbf{D}_2$. Also, if the rod is twisted about its axis or bent about an axis of symmetry of the cross-section, the response function ψ must be the same whatever the signs of the torsion or flexure. Hence ψ is form-invariant under a change of sign of the quantities γ_{12} , γ_{13} , γ_{23} , k_{12} , k_{13} , k_{21} , k_{23} , $D_{(1)}$, $D_{(2)}$, $D'_{(1)}$ and $D'_{(2)}$. From this invariance conditions it results for the response function ψ the following form

$$(2.1) \quad \begin{aligned} \lambda\psi = & A_{1111}\gamma_{11}^2 + 2A_{1122}\gamma_{11}\gamma_{22} + 4A_{1212}((\gamma_{12} + \gamma_{21})/2)^2 + A_{2222}\gamma_{22}^2 + \\ & + A_{1313}\gamma_{13}^2 + A_{2323}\gamma_{23}^2 + A_{1133}\gamma_{11}\gamma_{33} + A_{2233}\gamma_{22}\gamma_{33} + A_{3333}\gamma_{33}^2 + \\ & + B_{1111}\gamma_{11}k_{11} + B_{1122}\gamma_{11}k_{22} + B_{1212}k_{12}(\gamma_{12} + \gamma_{21}) + B_{1221}k_{21}(\gamma_{12} + \gamma_{21}) + \\ & + B_{1313}\gamma_{13}k_{13} + B_{2211}\gamma_{22}k_{11} + B_{2222}\gamma_{22}k_{22} + B_{2323}\gamma_{23}k_{23} + B_{1133}k_{11}\gamma_{33} + \\ & + B_{2233}k_{22}\gamma_{33} + C_{1111}k_{11}^2 + 2C_{1122}k_{11}k_{22} + C_{1212}k_{12}^2 + 2C_{1221}k_{12}k_{21} + \\ & + C_{1313}k_{13}^2 + C_{1231}k_{21}^2 + C_{2222}k_{22}^2 + C_{2323}k_{23}^2 - E_{11}\gamma_{11}\theta - E_{22}\gamma_{22}\theta - \\ & - E_{33}\gamma_{33}\theta - F_{11}k_{11}\theta - F_{22}k_{22}\theta - (G\theta^2/2). \end{aligned}$$

Thus, from 92 constitutive coefficients in (1.3) only 33 rest in (2.1). Also, in order to have the above symmetries, the constitutive coefficients must be functions on A_{33} , $D_{(1)}^2$, $D_{(2)}^2$, D_1 , $D'_{(1)}$, $D_{(2)}$, $D'_{(2)}$, $D_{(1)}^2$, $D_{(2)}^2$ and T . Further, we suppose ψ be form-invariant if ζ , $\mathbf{a}_3$, $\mathbf{A}_3$ change the sign. By imposing this change $k_{\alpha\beta}$, θ , and T no change the sign and $\gamma_{\alpha\beta}$ change the sign. Then the corresponding coefficients for $\gamma_{\alpha\beta}$ are linear functions of the quantities which change the sign from the assumed variables. So that we have

$$(2.2) \quad \begin{cases} B_{1313} = B'_{1313} D_{(1)} D'_{(1)} + B''_{1313} D_{(2)} D'_{(2)}, \\ B_{2323} = B'_{2323} D_{(1)} D'_{(1)} + B''_{2323} D_{(2)} D'_{(2)}. \end{cases}$$

The new coefficients (accent) and the other coefficients in (2.1) are functions of A_{33} , $D_{(1)}^2$, $D_{(2)}^2$, $D_{(1)}'^2$, $D_{(2)}'^2$, $D_1 D_2 D'_{(1)} D'_{(2)}$ and T .

For simplicity we introduce the following variables

$$(2.3) \quad \begin{cases} \hat{u}_i = u_i, \quad \hat{\delta}_{\alpha i} = (\delta_{\alpha i} / D_{(\alpha)}), \quad \hat{\delta}_{\alpha 3} = (\partial \hat{u}_\alpha / \partial \xi), \\ \hat{\delta}_{33} = (\partial \hat{u}_3 / \partial \xi) - (A'_{33} / 2 A_{33}) \hat{u}_{33}, \quad \hat{\theta} = (\theta / T); \end{cases}$$

$$(2.4) \quad \begin{cases} \hat{\gamma}_{\alpha\beta} = (\gamma_{\alpha\beta} / D_{(\alpha)} D_{(\beta)}) = \hat{\delta}_{\alpha\beta} + \hat{\delta}_{\beta\alpha}, \quad \hat{\gamma}_{\alpha 3} = (\gamma_{\alpha 3} / D_{(\alpha)}) = \hat{\delta}_{\alpha 3} + (\partial \hat{u}_\alpha / \partial \xi), \\ \hat{\gamma}_{33} = \gamma_{33} = 2((\partial \hat{u}_3 / \partial \xi) - (A'_{33} / 2 A_{33}) \hat{u}_{33}), \\ \hat{k}_{\alpha\beta} = (k_{\alpha\beta} / D_{(\alpha)} D_{(\beta)}) - D'_{(\alpha)} (\gamma_{\alpha\beta} / D_{(\alpha)}^2 D_{(\beta)}) = (\partial \hat{\delta}_{\alpha\beta} / \partial \xi), \\ \hat{k}_{\alpha 3} = (k_{\alpha 3} / D_{(\alpha)}) - D'_{(\alpha)} (\gamma_{\alpha 3} / D_{(\alpha)}^2) = \partial \hat{\delta}_{\alpha 3} / \partial \xi; \end{cases}$$

$$(2.5) \quad \hat{n}^i = n^i, \quad \hat{p}^{\alpha i} = D_{(\alpha)} p^{\alpha i}, \quad \hat{\pi}^{\alpha i} = D_{(\alpha)} \pi^{\alpha i} + D'_{(\alpha)} p^{\alpha i}, \quad \hat{\eta} = (\eta / T),$$

$$(2.6) \quad \hat{n}^\alpha = \hat{\pi}^{\alpha 3}, \quad \hat{\pi}^{\alpha\beta} = \hat{\pi}^{\beta\alpha}, \quad \hat{l}^{\alpha i} = D_{(\alpha)} l^{\alpha i}, \quad \hat{r} = (r / T).$$

The motion equations become

$$(2.7) \quad \begin{cases} \partial \hat{n}^\alpha / \partial \xi + \lambda \hat{l}^\alpha = \lambda \hat{u}^\alpha, \quad \partial \hat{n}^3 / \partial \xi + (A'_{33} / 2 A_{33}) \hat{n}^3 + \lambda \hat{l}^3 = \lambda \hat{u}^3, \\ \partial \hat{p}^{\alpha\beta} / \partial \xi + \lambda \hat{l}^{\alpha\beta} = \hat{\pi}^{\alpha\beta} + J^{\alpha\gamma} \hat{\delta}^{\gamma\beta}, \quad \partial \hat{p}^{\alpha 3} / \partial \xi + A'_{33} / 2 A_{33} \hat{p}^{\alpha 3} + \lambda \hat{l}^{\alpha 3} = \hat{\pi}^{\alpha 3} + J^{\alpha\gamma} \hat{\delta}^{\gamma 3}, \\ \partial (k \hat{\theta}) / \partial \xi + \lambda \hat{r} = \lambda \hat{\eta}. \end{cases}$$

The constitutive equations in terms of these variables have the form

$$(2.8) \quad \begin{cases} \lambda \hat{\psi}(\gamma_{ij}, k_{\alpha i}, D_{(\alpha)}, D'_{(\alpha)}) = \lambda \hat{\psi}(\hat{\gamma}_{ij}, \hat{k}_{\alpha i}, D_{(\alpha)}, D'_{(\alpha)}), \\ \hat{n}^\alpha = \lambda (\partial \hat{\psi} / \partial \hat{\gamma}_{\alpha 3}), \quad \hat{n}^3 = 2\lambda (\partial \hat{\psi} / \partial \hat{\gamma}_{33}), \quad \hat{\pi}^{\alpha\beta} = 2\lambda (\partial \hat{\psi} / \partial \hat{\gamma}_{\alpha\beta}), \quad \hat{p}^{\alpha i} = \lambda (\partial \hat{\psi} / \partial \hat{k}_{\alpha i}), \\ \hat{\eta} = -(\partial \hat{\psi} / \partial \hat{\theta}). \end{cases}$$

By replacing (2.4) in (2.1) we get for $\lambda \hat{\psi}$ the same expression (2.1) where the new coefficients are given by

$$(2.9) \quad \begin{cases} \hat{A}_{1111} = A_{1111} D_{(1)}^4 + B_{1111} D_{(1)}^3 D'_{(1)} + C_{1111} D_{(1)}^2 D_{(1)}', \\ 2\hat{A}_{1122} = 2A_{1122} D_{(1)}^2 D_{(2)}^2 + B_{1122} D_{(1)}^2 D'_{(2)} D_{(2)} + B_{2211} D_{(2)}^2 D_{(1)} D'_{(1)} + \\ + 2C_{1212} D_{(1)} D_{(2)} D'_{(1)} D'_{(2)}, \\ 4\hat{A}_{1212} = 4A_{1212} D_{(1)}^2 D_{(2)}^2 + 2B_{1212} D_{(2)}^2 D_{(1)} D'_{(1)} + 2B_{1221} D_{(1)}^2 D_{(2)} D'_{(2)} + \end{cases}$$

$$\begin{aligned}
& +2C_{1212}D_{(1)}^2D_{(2)}^2+2C_{1221}D_{(2)}^2D_{(1)}^2D_{(2)}^2+C_{2121}D_{(2)}^2D_{(2)}^2, \\
& \hat{A}_{1133}=A_{1133}D_{(1)}^2+B_{1133}D_{(1)}D_{(1)}^2, \quad \hat{A}_{2233}=A_{2233}D_{(2)}^2+B_{2233}D_{(2)}D_{(2)}^2, \\
& \hat{A}_{2222}=A_{2222}D_{(2)}^4+B_{2222}D_{(2)}^3D_{(2)}^2+C_{2222}D_{(2)}^2D_{(2)}^2, \\
& \hat{A}_{1313}=A_{1313}D_{(1)}^2+B_{1313}D_{(1)}D_{(1)}^2+C_{1313}D_{(1)}^2, \\
& \hat{A}_{2323}=A_{2323}D_{(2)}^2+B_{2323}D_{(2)}D_{(2)}^2+C_{2323}D_{(2)}^2, \quad \hat{A}_{3333}=A_{3333}, \\
& \hat{B}_{1111}=B_{1111}D_{(1)}^4+2B_{1133}D_{(1)}^3D_{(1)}^2, \\
& \hat{B}_{1122}=B_{1122}D_{(1)}^2D_{(2)}^2+2C_{1122}D_{(2)}^2D_{(1)}D_{(1)}^2, \\
& \hat{B}_{2211}=B_{2211}D_{(2)}^2D_{(1)}^2+2C_{1122}D_{(1)}^2D_{(2)}D_{(2)}^2, \\
& \hat{B}_{1212}=B_{1212}D_{(2)}^2D_{(1)}^2, \quad \hat{B}_{1221}=B_{1221}D_{(1)}^2D_{(2)}^2, \quad \hat{B}_{1133}=B_{1133}D_{(1)}^2, \\
& \hat{B}_{2222}=B_{2222}D_{(2)}^4+2C_{2222}D_{(2)}^3D_{(2)}^2, \quad \hat{B}_{2233}=B_{2233}D_{(2)}^2, \\
& \hat{B}_{1313}=B_{1313}D_{(1)}^2+2C_{1313}D_{(1)}D_{(1)}^2, \quad \hat{B}_{2323}=B_{2323}D_{(2)}^2+2C_{2323}D_{(2)}D_{(2)}^2, \\
& \hat{C}_{1111}=C_{1111}D_{(1)}^4, \quad 2\hat{C}_{1122}=2C_{1122}D_{(1)}^2D_{(2)}^2, \quad \hat{C}_{1212}=C_{1212}D_{(1)}^2D_{(2)}^2, \\
& 2\hat{C}_{1221}=2C_{1221}D_{(1)}^2D_{(2)}^2, \quad \hat{C}_{2121}=C_{2121}D_{(2)}^2D_{(1)}^2, \quad \hat{C}_{2222}=C_{2222}D_{(2)}^4, \\
& \hat{C}_{1313}=C_{1313}D_{(1)}^2, \quad \hat{C}_{2323}=C_{2323}D_{(2)}^2, \\
& \hat{E}_{11}=T(E_{11}D_{(1)}^2+F_{11}D_{(1)}^2), \quad \hat{E}_{22}=T(E_{22}D_{(2)}^2+F_{22}D_{(2)}^2), \quad \hat{E}_{33}=TE_{33}, \\
& \hat{F}_{11}=D_{(1)}^2TF_{11}, \quad \hat{F}_{22}=F_{22}D_{(2)}^2T, \quad \hat{G}=GT^2.
\end{aligned}
\tag{2.9}$$

From the constitutive equations (2.8) we get

$$\begin{aligned}
& \hat{n}^3=2\lambda(\partial\hat{\psi}/\partial\gamma_{33})=2(\hat{A}_{1133}\hat{\gamma}_{11}+\hat{A}_{2233}\hat{\gamma}_{22}+\hat{B}_{1133}\hat{k}_{11}+\hat{B}_{2233}\hat{k}_{22}-\hat{E}_{33}\hat{\theta}), \\
& \hat{n}^1=\hat{\pi}^{13}=\lambda(\partial\hat{\psi}/\partial\gamma_{13})=2\hat{A}_{1313}\hat{\gamma}_{13}+\hat{B}_{1313}\hat{k}_{13}, \\
& \hat{n}^2=\hat{\pi}^{23}=\lambda(\partial\hat{\psi}/\partial\gamma_{23})=2\hat{A}_{2323}\hat{\gamma}_{23}+\hat{B}_{2323}\hat{k}_{23}, \\
& \hat{\pi}^{11}=2\lambda(\partial\hat{\psi}/\partial\hat{\gamma}_{11})=2(2\hat{A}_{1111}\hat{\gamma}_{11}+2\hat{A}_{1212}\hat{\gamma}_{22}+\hat{A}_{1133}\hat{\gamma}_{33}+\hat{B}_{1111}\hat{k}_{11}+ \\
& \quad +\hat{B}_{1122}\hat{k}_{22}-\hat{E}_{11}\hat{\theta}), \\
& \hat{\pi}^{12}=2\lambda(\partial\hat{\psi}/\partial\hat{\gamma}_{12})=2(8\hat{A}_{1212}\hat{\gamma}_{12}+2\hat{B}_{1212}\hat{k}_{12}+2\hat{B}_{1212}\hat{k}_{21})=\hat{\pi}^{21},
\end{aligned}
\tag{2.10}$$

$$\begin{aligned}
 \hat{\pi}^{22} &= 2\lambda(\partial\hat{\psi}/\partial\hat{\gamma}_{22}) = 2(2\hat{A}_{2222}\hat{\gamma}_{11} + 2\hat{A}_{1122}\hat{\gamma}_{22} + \hat{A}_{2233}\hat{\gamma}_{33} + \hat{B}_{2211}\hat{k}_{11} + \\
 &\quad + \hat{B}_{2222}\hat{k}_{22} - \hat{E}_{22}\hat{\theta}), \\
 \hat{p}^{11} &= \lambda(\partial\hat{\psi}/\partial\hat{k}_{11}) = \hat{B}_{1111}\hat{\gamma}_{11} + \hat{B}_{2211}\hat{\gamma}_{22} + \hat{B}_{1133}\hat{\gamma}_{33} + 2\hat{C}_{1111}\hat{k}_{11} + \\
 &\quad + 2\hat{C}_{1122}\hat{k}_{22} - \hat{F}_{11}\hat{\theta}, \\
 \hat{p}^{12} &= \lambda(\partial\hat{\psi}/\partial\hat{k}_{12}) = 2\hat{B}_{1212}\hat{\gamma}_{12} + 2\hat{C}_{1212}\hat{k}_{12} + \hat{C}_{1321}\hat{k}_{21}, \\
 \hat{p}^{21} &= \lambda(\partial\hat{\psi}/\partial\hat{k}_{21}) = 2\hat{B}_{1221}\hat{\gamma}_{12} + 2\hat{C}_{1221}\hat{k}_{12} + 2\hat{C}_{2121}\hat{k}_{21}, \\
 \hat{p}^{22} &= \lambda(\partial\hat{\psi}/\partial\hat{k}_{22}) = \hat{B}_{1122}\hat{\gamma}_{11} + \hat{B}_{2222}\hat{\gamma}_{22} + \hat{B}_{2233}\hat{\gamma}_{33} + 2\hat{C}_{1122}\hat{k}_{11} + \\
 &\quad + 2\hat{C}_{2222}\hat{k}_{22} - \hat{F}_{22}\hat{\theta}, \\
 \hat{p}^{13} &= \lambda(\partial\hat{\psi}/\partial\hat{k}_{13}) = \hat{B}_{1313}\hat{\gamma}_{13} + 2\hat{C}_{1313}\hat{k}_{13}, \\
 \hat{p}^{23} &= \lambda(\partial\hat{\psi}/\partial\hat{k}_{23}) = \hat{B}_{2323}\hat{\gamma}_{23} + 2\hat{C}_{2323}\hat{k}_{23}, \\
 \lambda\hat{\gamma}_i &= -\lambda(\partial\hat{\psi}/\partial\hat{\theta}) = \hat{E}_{11}\hat{\gamma}_{11} + \hat{E}_{22}\hat{\gamma}_{22} + \hat{E}_{33}\hat{\gamma}_{33} + \hat{F}_{11}\hat{k}_{11} + \hat{F}_{22}\hat{k}_{22} + \hat{G}\hat{\theta}.
 \end{aligned}
 \tag{2.10}$$

Theorem. If the coefficients of the inertia tensor J^{21} , J^{21} vanish, then the motion equations separate in four sets.

Proof. By replacing (2.10) in (2.7) we get the displacement motion equations:

$$\begin{aligned}
 &[2\hat{A}_{1313}(\hat{\delta}_{13} + \hat{u}_1) + \hat{B}_{1313}\hat{\delta}'_{13}] + \lambda f^1 = \lambda \hat{u}_1, \\
 &[2\hat{A}_{2323}(\hat{\delta}_{23} + \hat{u}_2) + \hat{B}_{2323}\hat{\delta}'_{23}] + \lambda f^2 = \lambda \hat{u}_2, \\
 &2[2\hat{A}_{1133}\hat{\delta}_{11} + 2\hat{A}_{2233}\hat{\delta}_{22} + \hat{B}_{1133}\hat{\delta}'_{11} + \hat{B}_{2233}\hat{\delta}'_{22} - \hat{E}_{23}\hat{\theta}] + \\
 &+ (\hat{A}'_{33}/2\hat{A}_{33})(2\hat{A}_{1133}\hat{\delta}_{11} + 2\hat{A}_{2233}\hat{\delta}_{22} + \hat{B}_{1133}\hat{\delta}'_{11} + \hat{B}_{2233}\hat{\delta}'_{22} - \hat{E}_{23}\hat{\theta}) + \\
 &\quad + \lambda f^3 = \lambda \hat{u}_3, \\
 &[2\hat{B}_{1111}\hat{\delta}_{11} + 2\hat{B}_{2211}\hat{\delta}_{22} + 2\hat{B}_{1133}(\hat{u}_3 - (\hat{A}'_{33}/2\hat{A}_{33})\hat{u}_3) + \hat{C}_{1111}\hat{\delta}'_{11} + \\
 &\quad + 2(\hat{C}_{1122}\hat{\delta}'_{22} - \hat{F}_{11}\hat{\theta})] - 2[4\hat{A}_{1111}\hat{\delta}_{11} + 4\hat{A}_{1212}\hat{\delta}_{22} + \\
 &\quad + 2\hat{A}_{1133}(\hat{u}_3 - (\hat{A}'_{33}/2\hat{A}_{33})\hat{u}_3) + \hat{B}_{1111}\hat{\delta}_{11} + \hat{B}_{1122}\hat{\delta}'_{22} - \hat{E}_{11}\hat{\theta}] + \lambda l^1 = \\
 &\quad = J^{11}\ddot{\delta}_{11} + J^{12}\ddot{\delta}_{22}, \\
 &[2\hat{B}_{1122}(\hat{\delta}_{12} + \hat{\delta}_{21}) + 2\hat{C}_{1212}\hat{\delta}'_{12} + 2\hat{C}_{1221}\hat{\delta}'_{21}] - 4[4\hat{A}_{1212}(\hat{\delta}_{12} + \hat{\delta}_{21}) + \\
 &\quad + \hat{B}_{1212}\hat{\delta}_{12} + \hat{B}_{1221}\hat{\delta}_{21}] + \lambda l^{11} = J^{11}\ddot{\delta}_{12} + J^{12}\ddot{\delta}_{22},
 \end{aligned}
 \tag{2.11}$$

$$\begin{aligned}
 (2.11) \quad & [2\hat{B}_{1221}(\hat{\delta}_{12} + \hat{\delta}_{21}) + 2\hat{C}_{1221}\hat{\delta}'_{12} + 2\hat{C}_{2121}\hat{\delta}'_{21}]' - 4[4\hat{A}_{1212}(\hat{\delta}_{12} + \hat{\delta}_{21}) + \\
 & + \hat{B}_{1212}\hat{\delta}'_{12} + \hat{B}_{1221}\hat{\delta}'_{21}] + \lambda\hat{l}^{21} = J^{21}\ddot{\delta}_{11} + J^{22}\ddot{\delta}_{21}, \\
 & [2\hat{B}_{1222}\hat{\delta}_{11} + 2\hat{B}_{2222}\hat{\delta}_{22} + 2\hat{B}_{2233}(\hat{u}_3 - (A'_{33}/2A_{33})\hat{u}_3) + 2\hat{C}_{1122}\hat{\delta}'_{11} + \\
 & + 2\hat{C}_{2222}\hat{\delta}'_{22} - \hat{F}_{22}\hat{\theta}]' - 2[4\hat{A}_{2222}\hat{\delta}_{22} + 4\hat{A}_{1122}\hat{\delta}_{22} + 2\hat{A}_{2233}(\hat{u}_3 - \\
 & - (A'_{33}/2A_{33})\hat{u}_3) + \hat{B}_{2211}\hat{\delta}_{11} + \hat{B}_{2222}\hat{\delta}_{22} - \hat{E}_{22}\hat{\theta}] + \lambda\hat{l}^{22} = J^{21}\ddot{\delta}_{12} + J^{22}\ddot{\delta}_{22}, \\
 & [\hat{B}_{1313}(\hat{\delta}_{13} + \hat{u}_1) + 2\hat{C}_{1313}\hat{\delta}'_{13}] + (A'_{33}/2A_{33})[\hat{B}_{1313}(\hat{\delta}_{13} + \hat{u}_1) + 2\hat{C}_{1313}\hat{\delta}'_{13}] - \\
 & - 2\hat{A}_{1313}(\hat{\delta}_{13} + \hat{u}_1) + \hat{B}_{1313}\hat{\delta}'_{13} + \lambda\hat{l}^{13} = J^{11}\ddot{\delta}_{13} + J^{12}\ddot{\delta}_{23}, \\
 & [\hat{B}_{2323}(\hat{\delta}_{23} + \hat{u}_2) + 2\hat{C}_{2323}\hat{\delta}'_{23}] + (A'_{33}/2A_{33})[\hat{B}_{2323}(\hat{\delta}_{23} + \hat{u}_2) + 2\hat{C}_{2323}\hat{\delta}'_{23}] - \\
 & - [2\hat{A}_{2323}(\hat{\delta}_{23} + \hat{u}_2) + \hat{B}_{2323}\hat{\delta}'_{23}] + \lambda\hat{l}^{23} = J^{21}\ddot{\delta}_{13} + J^{22}\ddot{\delta}_{23}, \\
 & [k\hat{\theta}]' + \lambda\hat{r} = 2\hat{E}_{11}\hat{\delta}_{11} + 2\hat{E}_{22}\hat{\delta}_{22} + 2\hat{E}_{33}(\hat{u}_3 - (A'_{33}/2A_{33})\hat{u}_3) + \\
 & + \hat{F}_{11}\hat{\delta}'_{11} + \hat{F}_{22}\hat{\delta}'_{22} + \hat{G}\hat{\theta}.
 \end{aligned}$$

If the coefficients J^{12} , J^{21} vanish then the system (2.11) may be separated in the following four sets:

- i) the equations (2.11)_{1,8} for the unknowns $\hat{u}_1$ and $\hat{\delta}_{13}$ — the first flexion;
- ii) the equations (2.11)_{2,9} for the unknowns $\hat{u}_2$ and $\hat{\delta}_{23}$ — the second flexion;
- iii) the equations (2.11)_{5,6} for the unknowns $\hat{\delta}_{12}$ and $\hat{\delta}_{21}$ — the torsion;
- iv) the equations (2.11)_{3,4,7,10} for the unknowns $\hat{\delta}_{11}$, $\hat{\delta}_{22}$, $\hat{u}_3$ and $\hat{\theta}$ — thermo-extension.

Finally, we observe that in this case only the last problem differs from the isothermal problems.

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SIMETRII ÎN TERMOELASTICITATEA LINIARĂ A BARELOR CU DIRECTORI

(Rezumat)

Se consideră o bară dreaptă în configurația de referință și se presupune o serie de condiții de simetrie materială și geometrică. Se deduce forma funcției energie de deformare în urma acestor restricții. Cu ajutorul acestei funcții se scriu ecuațiile de mișcare ale teoriei. Se studiază cazul când sistemul de ecuații se separă în patru sisteme independente.

SINTETUL ÎN TERMOPLASTICITATEA LINIARĂ A BARELOR CU DIFECȚORI

(Rezumat)

Se consideră o bară direcțională în configurația de referință și se presupune o serie de condiții de simetrie materială și geometrică. Se deduc forme funcționale de deformare în urma acestor restricții. Cu ajutorul acestor funcții se scriu ecuațiile de mișcare ale barei. Se studiază cazul când starea de ecuații se separă în două sisteme independente.

LA SUPERGRAVITATION $N=1$ ET LE TRASPORT PARALLÈLE DE LEVI-CIVITA DU PLAN DE LOBACEVSKI

PAR

MARCEL AGOP et GH. ZET

On montre que la symétrie du Barbilian peut être acceptée dans la supergravitation $N=1$ pour le potentiel plan si on admet le transport parallèle de Levi-Civita du plan de Lobacevski.

Le groupe du Barbilian est le groupe homographique [1], [4]

$$(1) \quad k' = \frac{ah+b}{ch+d},$$

$$(2) \quad k' = \frac{\bar{c}h+d}{ch+d}k, \quad a, b, c, d \in \mathbb{R},$$

$$(3) \quad k = e^{i\varphi}, \quad \bar{h} = u + iv; \quad h = u - iv; \quad i = \sqrt{-1},$$

qui laissent invariant les 1-formes

$$(4) \quad \omega_0 = i \left(\frac{dk}{k} - \frac{dh + d\bar{h}}{h - \bar{h}} \right)^2,$$

$$(5) \quad \omega_1 = \omega_2 = \frac{dh}{k(h - \bar{h})}$$

et la 2-forme

$$(6) \quad \frac{ds^2}{g} = - \left(\frac{dk}{k} - \frac{dh + d\bar{h}}{h - \bar{h}} \right)^2 + 4 \frac{dh d\bar{h}}{(h - \bar{h})^2}; \quad g = \text{const.}$$

Si est remplie la relation

$$(7) \quad d\varphi = \frac{du}{v}$$

alors

$$(8) \quad \frac{ds^2}{g} = 4 \frac{dh d\bar{h}}{(h - \bar{h})^2} = \frac{du^2 + dv^2}{v^2}$$

et φ défini le transport parallèle de Levi-Civita du plan de Lobacevski pour la métrique (6). Pour montrer il est nécessaire d'observer que la métrique (6) est conforme euclidienne avec la facteur de conformité

$$(9) \quad E(u, v) = \frac{1}{v^2}.$$

Dans ces conditions on peut calculer l'angle entre le vecteur initial et celui transporté par parallélisme comme intégrale de l'équation [2]

$$(10) \quad d\varphi = -\frac{1}{2} \left(\frac{\partial \ln E}{\partial v} du - \frac{\partial \ln E}{\partial u} dv \right)$$

le long d'une courbe de transport. En tenant compte de (8) et (9) dans (10) on obtient (7).

Par la submerssion de la métrique (8) sur une variété minkowskienne [3] ou obtient l'invariant

$$(11) \quad L = g \frac{\partial_\mu h \partial_\mu \bar{h}}{(h - \bar{h})^2}; \quad \partial_\mu = \frac{\partial}{\partial x^\mu}.$$

Cet invariant peut jouer le rôle de fonction Lagrange de la supergravitation $N=1$ pour le potentiel plan si $g=3$ et h a la signification de superchamp scalaire [5], [6], [7]. Ainsi on peut admettre le groupe du Barbilian comme un groupe de symétrie dans la supergravitation $N=1$ si on impose le transport parallèle de Levi-Civita du plan de Lobacevski.

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SUPERGRAVITAȚIA $N=1$ ȘI TRANSPORTUL PARALEL LEVI-CIVITA AL PLANULUI LOBACEVSKI

(Rezumat)

Se arată că simetria Barbilian poate fi acceptată în supergravitația $N=1$ cu potențial plan dacă se admite transportul paralel Levi-Civita al planului Lobacevski.

LE PARALLÉLISME DES DIRECTIONS ET LES COORDONNÉES PHYSIQUES

PAR

GH. ZET, MARCEL AGOP, MARINA DARIESCU et VALENTINA SÂNDULESCU

On montre que le transport parallèle de Levi-Civita de plan du Lobachevski est transport des directions. Les coordonnées compatibles avec ce transport sont des coordonnées d'épave.

La physique actuelle, dans la mesure où la relativité générale est impliquée, est redevable aux mathématiques par la nécessité de définir un parallélisme. Il existe là plusieurs types de parallélisme testable phénoménologiquement: Levi-Civita, Fermi-Walker, etc. Tous ces types de parallélisme supposent le calcul différentiel de deux voisinages spatiales infinitésimales, ce qui implique une certaine échelle à considérer l'univers.

Phénoménologiquement, les mesures passent sur cette échelle: il est possible (et, parfois, il est nécessaire) d'utiliser pour les caractérisations macroscopiques, des phénomènes essentiellement microscopiques.

De [1] il résulte que la notion transcende éventuellement l'échelle de considération de l'univers est la notion de direction. C'est pourquoi on pose le problème d'une définition du parallélisme qui se rapporte exclusivement aux directions, c'est-à-dire qu'elle soit valable à toute échelle. Il est à prévoir, que de cette façon, le parallélisme ne soit plus tributaire à la géométrie différentielle.

La modalité la plus adéquate de définir un parallélisme doit être celle qui s'intéresse de près à la manière dont on fait les mesures géométriques (qui, on le sait, ne peuvent pas toujours donner la distance). Elle a été indiquée par Cartan [2], dont nous extrayons les traits essentiels. Ce qui est fondamental dans la définition du parallélisme selon Cartan c'est „la mémorisation“ des composantes d'un vecteur dans un repère orthogonale choisi arbitrairement et défini justement par rapport à ce vecteur. Le transport par le parallélisme dans un autre lieu que celui où est défini le repère cartésien signifie la reproduction du vecteur dans un repère cartésien lié à ce lieu.

Mais en ce qui concerne la position, la notion de vecteur implique celle de distance qui, ne peut pas être définie exactement (par des mesures) que dans les cas particuliers et, de la même manière, les composantes cartésiennes du vecteur.

Ce qui peut être „mémorisé“ toujours c'est la direction rendue par les deux angles sphériques θ et φ , qui peuvent se trouver localement, par l'orien-

tation d'un appareil théodolique. Le point de vue géométrique local est ici exprimé par le fait que les directions des diverses repères qui ont les mêmes paramètres (θ, φ) sont, en fait, l'une et la même direction. Elles donnent la classe d'équivalence caractérisée quantitativement par les paramètres (θ, φ) , exactement comment dans la géométrie euclidienne habituelle, on définit la classe d'équivalence pour un vecteur donné.

En acceptant, par exemple, un point de vue, par le repère O , les directions de la classe d'équivalence par le parallélisme diffèrent entre elles et on pose le problème de la paramétrisation de cette différence. Il faut mentionner dans ce cas que le point de vue global est marqué d'une grande arbitrarité: on peut choisir comme repère global a priori tout repère, non pas seulement O . Par conséquent, la paramétrisation sera elle-aussi marquée par cette arbitrarité.

Dans la Fig. 1 nous donnons, pour fixer les idées, une multitude de repères différant par leur origine et par l'une des axes. La figure représente une classe d'équivalence par le parallélisme qui représente la direction (θ, φ) .

Notre avis est que la plus adéquate mise en équation de ce problème est facilitée par la représentation de la direction (θ, φ) par la matrice [1]

$$(1) \quad \hat{I} = \begin{pmatrix} \frac{h+\bar{h}}{2} & -h\bar{h} \\ 1 & -\frac{h+\bar{h}}{2} \end{pmatrix},$$

qui représente, comme homographie, une involution elliptique.

La métrique cayleyenne de cette involution a jusqu'à une constante arbitraire, la forme [1]

$$(2) \quad -\frac{ds^2}{g} = \frac{dh d\bar{h}}{(h-\bar{h})^2},$$

résultat qui indique le fait que, en vérité, le transport parallèle défini dans [1] est un transport de directions.

La même involution elliptique (1) nous permet de départager aussi les divers points de vues globales par deux paramètres réels, ou, du point de vue projectif, par un seul paramètre, de la manière suivante: nous acceptons comme définitoires pour une direction donnée les points fixes de l'involution (1), en représentant globalement la classe d'équivalence comme une matrice aux mêmes points fixes. Comme nous savons [3] une telle matrice s'exprime comme une combinaison linéaire de la matrice unité et de la matrice (1) même

$$(3) \quad \hat{M} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} \frac{h+\bar{h}}{2} & h\bar{h} \\ 1 & -\frac{h+\bar{h}}{2} \end{pmatrix}.$$

Par conséquent, les divers points de vue globaux sont départagés par les diverses valeurs des paramètres λ et μ . Ces valeurs doivent caractériser uniquement le repère global choisi.

Maintenant, toute matrice de la forme (3) détermine un plan abstrait avec les matrices de base $\hat{J}$, $\hat{K}$, en traitant $\hat{J}$, $\hat{K}$ comme des matrices de décomposition de I . La forme des matrices $\hat{J}$, $\hat{K}$ peut être obtenue par la théorie générale [3] en décomposant (1) en un produit à deux involutions hyperboliques. Cela est réalisé, le plus simplement, si on part de la considération que les équations des points fixes des trois involutions doivent être trois quadriques mutuellement conjuguées, c'est-à-dire ces trois quadriques doivent avoir les racines deux par deux dans un rapport harmonique. Ayant en vue le fait que le polynôme des points fixes de $\hat{I}$ est

$$(4) \quad 2(x-u-iv)(x-u+iv),$$

les deux autres polynômes seront

$$(5) \quad 2(x^2-2ux+u^2-v^2)$$

et

$$(6) \quad 2(x-u).$$

Par conséquent, normées d'une manière adéquate, les involutions $\hat{J}$ et $\hat{K}$ ont la forme

$$(7) \quad J = \frac{i}{v} \begin{pmatrix} u & -u^2+v^2 \\ 1 & -u \end{pmatrix}$$

et

$$(8) \quad \hat{K} = i \begin{pmatrix} -1 & zu \\ 0 & 1 \end{pmatrix}.$$

Les relations

$$(9) \quad \Delta(\hat{I}) = \Delta(\hat{J}) = \Delta(\hat{K}) = 1, \\ \hat{I}^2 = \hat{J}^2 = \hat{K}^2 = -\hat{E}$$

sont satisfaites, le tableau de multiplication étant dicté dans ce cas par

$$(10) \quad \hat{I} \cdot \hat{J} = \hat{K},$$

Dans les relations (9) la matrice $\hat{E}$ représente la matrice unité.

Les polynômes caractéristiques (X, Y, Z) de ces trois involutions ont la forme

$$(11) \quad X = \frac{1}{v} [(x-u)^2 + v^2],$$

$$Y = \frac{i}{v} [(x-u)^2 + v^2],$$

$$\text{ou} \quad Z = zi(x-u),$$

$$h = u + iv$$

et ils satisfont la condition de Yamamoto [4]

$$(12) \quad X^2 + Y^2 + Z^2 = 0.$$

De la sorte, on peut associer un système de trois paramètres qui satisfait la condition (12) à n'importe quelle classe d'équivalence par parallélisme Cartan. Les trois paramètres peuvent avoir le rôle de coordonnées d'épave.

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PARALELISMUL DE DIRECȚII ȘI COORDONATELE FIZICE

(Rezumat)

Se arată că transportul paralel Levi-Civita al planului Lobacevski este un transport de direcții. Coordonatele compatibile cu acest transport sînt coordonatele de spin.

L'ÉTUDE DU COMPORTEMENT PIÉZOÉLECTRIQUE DES CÉRAMIQUES DE TYPE PZT MODIFIÉ PAR BI ET LI À LA VARIATION DE L'INTENSITÉ DU CHAMP ÉLECTRIQUE APPLIQUÉ

PAR

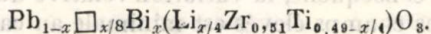
IRINA ANTONESCU, M. ANTONESCU, POMPILIU NICOLAU et C. TĂNĂSOIU

1. Introduction. Étant connue l'influence positive des modifiants qui créent des vacances sur les sites dodécaédriques sur la valeur du facteur de couplage planaire [1], on a réalisé un système de composés nouveaux, conçus avec une déficience initiale de plomb sur les sites dodécaédriques [2].

Leur étude complète a inclus la détermination du comportement piézoélectrique par rapport à la variation du champ électrique appliqué.

Vu que les céramiques piézoélectriques sont ferroélectriques, leur comportement général dans des champs électriques croissants est non-linéaire : au début est favorisée la réorientation des domaines ferroélectriques, par l'augmentation de leur degré d'alignement, pour qu'ensuite le matériau se voie dégradé. Par conséquent, pour chaque matériau doivent être précisées la valeur de l'intensité du champ à partir de laquelle le degré d'alignement commence à diminuer (la valeur du champ qui marque le début de la dégradation du matériau), tout comme la valeur maximale de la variation relative de l'activité piézoélectrique dans le domaine des valeurs de champ investiguées.

2. La méthode expérimentale. Le système étudié contient des composés non-stoechiométriques [2], avec des vacances cationiques de type A, dont la formule générale est



On l'a nommé BiLi. Les composés dont le nom est BiLi 1, BiLi 2 et, respectivement, BiLi3 correspondent aux valeurs suivantes de la variable x : 0,01 ; 0,02 et, respectivement, 0,03.

Les observations métallographiques ont mis en évidence des cristallites uniformes, dont les dimensions dépassent $5\text{ }\mu\text{m}$ au cas des composés BiLi 1 et BiLi 2 et sont moindres, sous $3\text{ }\mu\text{m}$, au cas du composé BiLi 3 [2].

L'étude du comportement piézoélectrique dans des champs dont la valeur a varié entre 0,04 et 1 kV/cm a été fait d'après une méthode originale [2]. Cette étude a compris la détermination des variations du facteur de couplage k_p et de la constante de la fréquence dans le mode radial de vibration N_p , par rapport aux variations de l'intensité du champ électrique appliqué à l'éprouvette.

En employant les données qui en résultent on a pu calculer les variations relatives de l'activité piézoélectrique par rapport à l'accroissement du champ et on a pu préciser les valeurs du champ à partir desquelles commence la dégradation, au cas de chaque matériau. En utilisant le développement selon les polynômes orthogonaux Tchébychev, assisté par calculateur, on a pu évaluer les coefficients des équations regressives $k_p(E)$ et $N_p(E)$ correspondantes à chaque matériau.

3. Résultats et discussions. Les variations du facteur de couplage k_p et de la constante de la fréquence N_p , obtenues expérimentalement, sont représentées dans les Figs. 1 et 2.

Les équations regressives qui décrivent le comportement piézoélectrique des composés du système BiLi dans des champs électriques élevés sont les suivantes :

$$k_p(E) = -0,236E^2 + 0,365E + 0,460 \quad \text{pour BiLi 1}$$

$$k_p(E) = -0,456E^2 + 0,466E + 0,584 \quad \text{pour BiLi 2}$$

$$k_p(E) = -0,265E^2 + 0,321E + 0,577 \quad \text{pour BiLi 3}$$

et respectivement

$$N_p(E) = +179,4E^2 - 384,4E + 2235 \quad \text{pour BiLi 1}$$

$$N_p(E) = +371,0E^2 - 480,0E + 2030 \quad \text{pour BiLi 2}$$

$$N_p(E) = +359,8E^2 - 471,5E + 2195 \quad \text{pour BiLi 3.}$$

On voit que la valeur maximale du facteur de couplage piézoélectrique est obtenue au cas d'une substitution de 2% ions de Bi sur les sites de type A. D'ailleurs le composé BiLi 2 est le plus maniable du point du vue technologique, se densifie facilement, à des températures relativement basses (1100°C), l'accroissement des cristallites est assez rapide, en obtenant des granulations uniformes, avec des cristallites de 5...10 μm .

L'augmentation du pourcentage du Bi et, par conséquent, des vacances sur les sites de type A, conduit à des composés (BiLi 3) dont la granulation est plus fine (cristallites dont les dimensions sont inférieures à 3 μm) et dont le facteur de couplage piézoélectrique est moins influençable par le champ électrique ; par conséquent, la variation relative du facteur de couplage piézoélectrique dans des champs élevés est minimale au cas du matériau BiLi 3 (17%). Vu la finesse de la texture piézoélectrique, la valeur du champ à partir duquel le degré d'alignement des domaines ferroélectriques commence à diminuer est plus grande au cas de BiLi 3 (0,6 kV/cm) qu'au cas de BiLi 2 (0,5 kV/cm).

Le matériau BiLi 1 a une moindre concentration de vacances sur les sites de type A et des cristallites d'environ 5 μm . On peut expliquer ainsi les petites valeurs du facteur de couplage piézoélectrique et, respectivement, la valeur élevée du champ à partir duquel commence la dégradation du matériau (0,8 kV/cm).

Les courbes qui donnent les lois de variation de la constante de la fréquence N_p par rapport à l'accroissement du champ électrique appliqué confirment tant

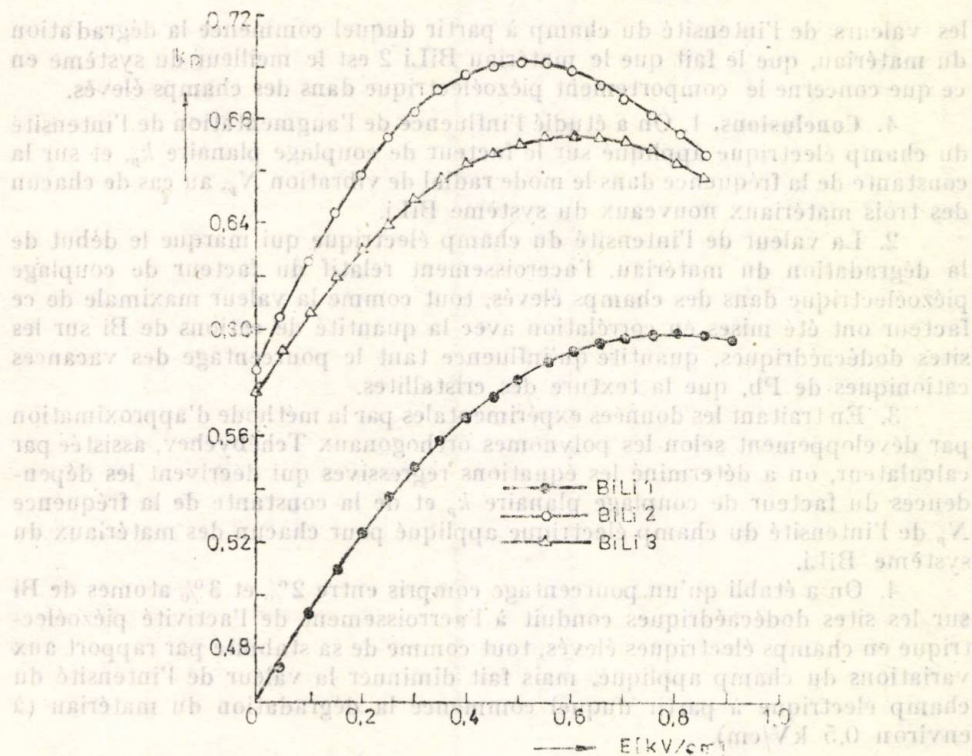


Fig. 1

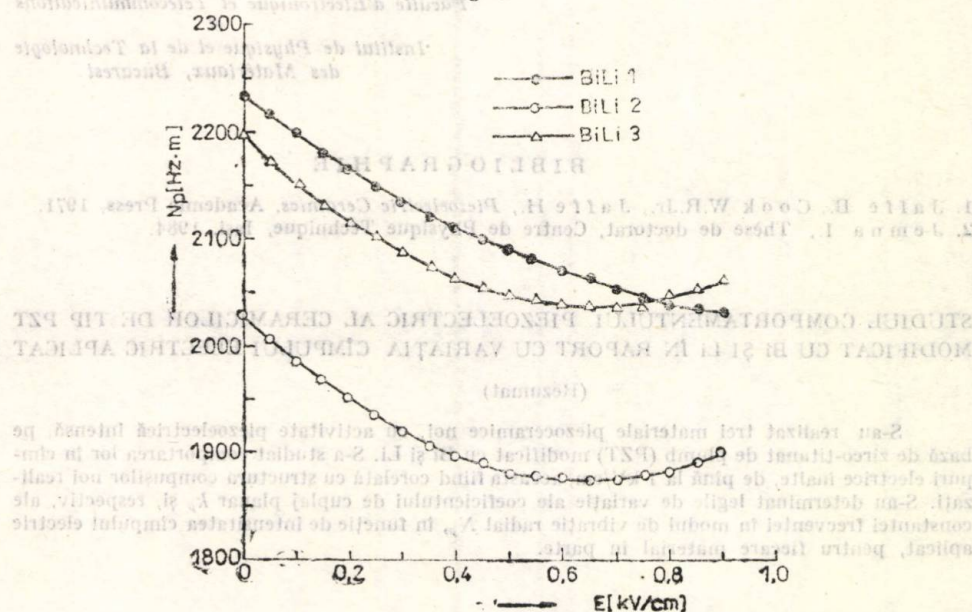


Fig. 2

les valeurs de l'intensité du champ à partir duquel commence la dégradation du matériau, que le fait que le matériau BiLi 2 est le meilleur du système en ce que concerne le comportement piézoélectrique dans des champs élevés.

4. Conclusions. 1. On a étudié l'influence de l'augmentation de l'intensité du champ électrique appliqué sur le facteur de couplage planaire k_p , et sur la constante de la fréquence dans le mode radial de vibration N_p , au cas de chacun des trois matériaux nouveaux du système BiLi.

2. La valeur de l'intensité du champ électrique qui marque le début de la dégradation du matériau, l'accroissement relatif du facteur de couplage piézoélectrique dans des champs élevés, tout comme la valeur maximale de ce facteur ont été mises en corrélation avec la quantité de cations de Bi sur les sites dodécaédriques, quantité qu'influence tant le pourcentage des vacances cationiques de Pb, que la texture des cristallites.

3. En traitant les données expérimentales par la méthode d'approximation par développement selon les polynômes orthogonaux Tchébychev, assistée par ordinateur, on a déterminé les équations regressives qui décrivent les dépendances du facteur de couplage planaire k_p et de la constante de la fréquence N_p de l'intensité du champ électrique appliqué pour chacun des matériaux du système BiLi.

4. On a établi qu'un pourcentage compris entre 2% et 3% atomes de Bi sur les sites dodécaédriques conduit à l'accroissement de l'activité piézoélectrique en champs électriques élevés, tout comme de sa stabilité par rapport aux variations du champ appliqué, mais fait diminuer la valeur de l'intensité du champ électrique à partir duquel commence la dégradation du matériau (à environ 0,5 kV/cm).

Reçu le 15 Décembre, 1989

Institut Polytechnique, Jassy,
Faculté d'Electronique et Télécommunications
et
Institut de Physique et de la Technologie
des Matériaux, Bucarest

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2. Jemna I., Thèse de doctorat, Centre de Physique Technique, Iași, 1984.

STUDIUL COMPORTAMENTULUI PIEZOELECTRIC AL CERAMICILOR DE TIP PZT MODIFICAT CU Bi ȘI Li ÎN RAPORT CU VARIAȚIA CÎMPULUI ELECTRIC APLICAT

(Rezumat)

S-au realizat trei materiale piezoceramice noi, cu activitate piezoelectrică intensă, pe bază de zirco-titanat de plumb (PZT) modificat cu Bi și Li. S-a studiat comportarea lor în cîmpuri electrice înalte, de pînă la 1 kV/cm, aceasta fiind corelată cu structura compușilor noi realizați. S-au determinat legile de variație ale coeficientului de cuplaj planar k_p și, respectiv, ale constantei frecvenței în modul de vibrație radial N_p , în funcție de intensitatea cîmpului electric aplicat, pentru fiecare material în parte.

Professor Dr. ȘTEFANIA RUSCIOR

(1919 — 1989)



Professor dr. Ștefania Ruscior, an outstanding member of the Mathematics Department of the „Gheorghe Asachi” Polytechnic Institute in Jassy, died on the 3 rd of september, 1989.

During a 37 years didactic and scientific research activity, professor Ștefania Ruscior brought a great contribution to the studies carried on by the school of mathematics in the Jassy university centre.

Professor Ștefania Ruscior was born on the 1 st january, 1919, in Bucharest, in the family of a distinguished superior officer, enjoying an excellent education during her girlhood.

Between 1929—1937 she attended secondary and high school at „Oltea Doamna”, the famous high school for girls in Jassy. For the rest of her lifetime she felt attached and grateful to that school and mainly to the teacher of mathematics there, prof. Silvia Creangă.

Later on, between 1937—1940 she studied mathematics at the Faculty of Sciences of the University „Al. I. Cuza” in Jassy graduating it with “magna cum laudae”. During the academic year 1940—1941 she went on improving her didactic and teaching practice at the University Pedagogical Seminary in Bucharest.

Beginning with 1942 she worked at the Department of Mathematics in Polytechnic Institute of Jassy, as an assistant professor (1942-1949), lecturer (1949—1951) and professor (1951—1979).

In order to better meet the specific requirements of a polytechnic institute and to fully understand her students work, as well as the technical sciences, she attended the Polytechnic Institute and, in 1948, became an engineer in electromechanics.

While at the Polytechnic Institute she gave courses of *Elementary Geometry* at the Faculty of Civil Engineering, *Analytic and Differential Geometry* at the Faculties of Mechanics and Electrotechnics, *Descrip-*

tive Geometry at the Faculties of Mechanics and Civil Engineering, *Higher Mathematics* at the Faculty of Industrial Chemistry.

She published, in co-operation, various textbooks for students: *Higher Mathematics* in 1969, a *Course on Differential Equations* in 1971, *Elements of the Theory of Functions of a Complex Variable*, also in 1971.

In 1961, prof. Ștefania Ruscior became doctor in Mathematics by submitting the valuable doctoral thesis *Correspondences through Parallel Tangent Linear Manifolds among Ruled Manifolds* under the guidance of Professor dr. Ion Creangă from the University of Jassy.

The scientific activity carried out by prof. Ștefania Ruscior was resulting in papers published both in the country and abroad, as well as in unpublished papers but communicated at various scientific meetings. She is the only author of more than 30 papers, most of them published or communicated in Italy, Belgium, Poland, Czechoslovakia.

Prof. Ștefania Ruscior was highly interested in the *correspondence through parallel tangent planes* (or hyperplanes), *between ruled surfaces* (or hypersurfaces) *in the space S_4* . She brought remarkable contributions to the study of the *correspondence through parallel tangent hyperplanes and partial parallel tangent hyperplanes between ruled manifolds in the space S_n* , of some networks on surfaces or hypersurfaces, of the algebraic aspects of the correspondence through parallelism of ruled manifolds in the space S_n . She obtain the projective and affine classification of the ruled manifolds in the space S_n , using the number of the singular points on a generic generatrix and the dimension of the improper section of the manifold. By using the modular geometric lattices and introducing a numerical function which characterizes the partial parallelism, she studied the algebraic aspects of the correspondence through parallelism. She also studied the networks with equal geodesic torsion in the space S_3 and extended the Myller networks on manifolds in the space S_4 .

Prof. Ștefania Ruscior took part in a lot of national and international scientific meetings (Poland, Czechoslovakia, Italy). All her achievements successfully carried on the traditions of the school of geometry in Jassy and our country.

Prof. Ștefania Ruscior gave a great contribution to solving problems concerning the department she worked in; she was an active member of different boards for analysing courses and seminars, for examining the candidats for a doctor's degree.

Prof. Ștefania Ruscior retired from her activity in 1979, after a fruitful didactic and scientific activity.

Her death has been a great loss for our department and for the higher education in Jassy, as well. The members of the Department of Mathematics will always recall her personality, her highly appreciated qualities as a teacher and scientist.

Department of Mathematics

RECENZII

BOOK REVIEWS

РЕЦЕНЗИИ

A. KANEKO, *Introduction to Hyperfunctions*. Kluwer Academic Publishers, 1988.

This book is an introduction to the generalized functions theory, called also hyperfunctions, of the Japanese mathematician M. Sato. In the presentation of the distributions theory, M. Sato has used another method, different of that used by L. Schwartz. It is well known that in Schwartz's theory, the test-functions play a central role. But these functions are not analytical, because of the compact support. M. Sato considers that from some viewpoint, this fact may be unsatisfactory and then he defines the distributions (hyperfunctions) using functions $F(z)$, holomorphic in the semiplanes $\Im z > 0$ and $\Im z < 0$, identifying a hyperfunction with the difference $F(x+i0) - F(x-i0)$. For example, we have for the Dirac distribution $\delta(x) = -(1/2\pi i)(1/(x+i0) - 1/(x-i0))$, consequently in this case $F(z) = -1/(2\pi iz)$. This definition permits elegant proofs and the development of a relatively simple calculus.

The author point out that, although this theory does not contains that of L. Schwartz as a particular case, it is very general, being less known only because that in contrast with the classical theory, there exist few books and manuals concerned to this subject. On the other hand, we must remark that for the understanding of this book, serious knowledges of the complex function theory of one and several variables and of some new chapters as topology, cohomological algebra, etc. are necessary. In the first part of the book, the author gives examples, explains the concept of hyperfunction, develops the calculus with hyperfunctions, while the second part is centered on the foundations of this theory. This fact is a major advantage for the readers, namely researchers and graduate students who are interested in the theory of functions of complex variable. The book contains many exercises and problems, related to the exposed theory. The final chapter presents the solutions of the proposed problems.

Finally, we remark that this excellent book is devoted to one of the most important contribution of the Japanese mathematics to the development of the world science.

Adrian Corduneanu

B.W. SCHULZE and H. TRIEBEL (Eds.), *Partial Differential Equations*. Teubner Texte zur Mathematik, Band 412, B.G. Teubner Verlagsgesellschaft, Leipzig, 1988.

This book, whose editors are Prof. B.W. Schulze (Berlin) and Prof. H. Triebel (Jena), begins with the list of lectures given at the Conference *Partial Differential Equations 25-29 April 1988*, held in Holzhausen, by mathematicians from some European countries. This volume also contains the list of the names and the addresses of all the participants to this scientific meeting.

The content consists by 27 works of some renowned participants at this Conference and of the other known specialists in the field. Of course, it is difficult to resume and to characterize in few words all these works, but we can say that they are concerned, in the most part, to some important problems in the theory of partial differential equations, such as the existence and the asymptotic behaviour of the solutions, the spectral theory of Schrödinger and other types of operators, the pseudodifferential operators, variational inequalities for nonlinear operators, applications in thermoelasticity, etc.

The editors announce that the following conference in this field will be given in 1992. Finally, we also mention the remarkable graphical presentation of this volume common to all the texts of the Teubner Collection.

Adrian Corduneanu

DONALD COLES (Ed.), **Perspectives in Fluid Mechanics**. Proceedings of a Symposium Held on the Occasion of the 70th Birthday of Hans Wolfgang Liepmann, Pasadena, California, 10–12 January 1985. *Lecture Notes in Physics* (Eds. H. Araki, J. Ehlers, K. Hepp, R. Kippenhahn, D. Ruelle, H.A. Weidenmüller, J. Wess, J. Zittartz), Vol. 320, Springer Verlag, ISBN 3-540-50644-6, Berlin–Heidelberg–New York–London–Paris–Tokyo, 1988, VII+207 pp.

This book records the proceedings of a symposium called *Perspectives in Fluid Mechanics* held at the California Institute of Technology in Pasadena on 10–12 January, 1985. The symposium was attended by more than 350 persons. Sixteen invited papers were presented.

The topics covered in this volume mainly reflect current interest and activity in fluid mechanics. This topic was chosen to celebrate the 70th birthday of Hans Wolfgang Liepmann, Charles Lee Powell Professor of Fluid Mechanics and Thermodynamics at California Institute of Technology.

The book begins with a brief dedication to Hans Wolfgang Liepmann from the part of the editor. It contains nine invited papers written by some of the leading scientists working in the field of fluid mechanics. All contributions have been updated for this edition and reflect Liepmann's wide interests in science.

Much of this volume (five papers) is devoted to the nonlinear physics of fluid turbulence, area in which Liepmann himself has worked. In his paper *The Rise and Fall of Ideas in Turbulence*, published ten years ago in *American Scientist*, Liepmann observes: "...turbulence is and will remain the most difficult problem of fluid mechanics". The papers collected here present the recent results of *experimental, theoretical and computational investigations of turbulence*. There are contributions on large-scale ocean turbulence, dynamics of vorticity, turbulence research by numerical simulation, the role of cartoons in turbulence and dynamical system theory. Of particular interest to me was the invited paper by Albert Libchaber on *dynamical system theory*. In this paper a critical review of the present state of knowledge of transition to turbulence is presented. The remaining four papers treat *transonic aerodynamics, astrophysical jets, human-powered flight and geologic nozzles*.

Exceptionally well written and well illustrated, the book includes an extensive reference list at the end of each paper, numerous clear diagrams and some splendid photographs. The material is presented in a didactic way, so as to be useful to all researchers in the field.

This is an excellent book and I can strongly recommend it not only to graduate students and researchers in fluid mechanics but also to all those interested in the fascinating but difficult problem of turbulence.

Viorel Stancu

D. CARTIANU, **Synchronization Phenomena in Reaction-Diffusion Systems**. *Teubner Texte zur Physik*, Band 8, BSB.B.G. Teubner Verlagsgesellschaft, Leipzig, 1986, ISBN 3-322-00334-5, 283 pp.

During the last two decades it was proved that dissipative structures have a lot of interesting behaviours. This is the reason for their recent intensive study (both experimentally and theoretically). One of the most important dissipative structures is the reaction-diffusion system. The present book analyses driven reaction-diffusion systems and contains many author's own results. The driven reaction-diffusion system as a dynamical system is studied by recent modern mathematical methods (bifurcation theory, multiple time scale method, Fredholm alternative theorem, invariant manifold theorem etc.). The book is the first one in this field and it could become an outstanding guide book.

For further information, the title of each chapter will be presented: 1. *General introduction*, 2. *Phase and Frequency Synchronization in Reaction-Diffusion Equation*. 3. *Dissipative Structures in Reaction-Diffusion Systems as Synchronization Phenomena*. 4. *The Improvement of the Ener-*

getic Efficiency of Chemical Processes in the Oscillatory Mode. 5. Cooperative Phenomena in Dissipative Systems Far from Thermodynamic Equilibrium. Bifurcation of Reaction-Diffusion Systems. 6. Synchronization Phenomena of Periodically Stimulated Dissipative System in Bifurcation Theory and in the Alternative Theory of Averaging. 7. Rate Constrained Chemical Equilibrium Method for Solving Complex Mass Action Kinetic Systems.

The book can be regarded as a monograph in a interdisciplinary field and is addressed equally to chemists, physicists, mathematicians, biologists and a large class of engineers. The young researchers could find interesting also ideas, for future work. I warmly recommend the book and I think that this it is a valuable acquisition for any academic library.

I. Grosu

K. SHIMODA, T. RYU (Eds.), **International Conference on Trends in Physics Education.** Sophia University, Tokyo, August 24–29, 1986. Published by The Physics Education Society of Japan, The International Commission of Physics Education, KTK Scientific Publishers, 1986, 482 pp.

The book begins with *Address to the International Conference on Trends in Physics Education* by R. Kubo (Nobel Prize). He suggests that new terms like instabilities, nonlinear dynamics and fractals must be considered by physics education. The next „Address“ is by S.M. Haggis (UNESCO) who recommends „making physics more enjoyable...“.

The Chapter „Plenary Lectures“ contains 15 lectures given by leading personalities in research and development of physics education as follows:

A. *Research in Physics Education and its Implementation in the Classroom*, by R. Driver (England), A. Tilberghien (France), Y. Kakiachi (Japan), K. Toh (Singapore), P. Black (England).

B. *The Impact of Developments in Technology on Classroom Teaching*, by J. Ogborn (England), E. Taylor (U.S.A.), Y. Yun (P.R. China), K. Hirata (Japan).

C. *The Motivation of Students Toward the Study of Physics*, by V. Talisayon (Philippines), G. Marx (Hungary), L. Lijnse (The Netherlands), M. Kondo (Japan).

D. *Special Lectures*, by M. Kikuchi (Japan), K. Kinoshita (Japan). These lectures contain a lot of important results obtained from research in physics education.

The Chapter „Workshops“ presents conclusions from the following workshops: *Evaluation of Physics Teaching; Learning Physics Concepts; New Approaches in Teacher Training; Teaching Physics in Secondary School; Low Cost Experiments and Demonstrations; New Topics in Physics Education; Physics with a Difference: the Plon Experience; New Course Materials at Secondary School Level; Electronics in Senior School Physics; New Course at Secondary School Level; Physics for all Students; Making Computational Models; Computer Assisted Physics Learning; The Microcomputer as a Laboratory Instrument; Physics and Art; The Mechanical Universe; Audio-visual Materials and other Application to Physics Schools; Physics and Society.*

The next chapter „Oral Sessions“ contains 103 papers grouped in the sections: *Learning Physics Concepts; Teaching Physics in Secondary Schools; Teaching Physics in Colleges and Universities; Laboratory Experiments and Demonstrations; The Use of Computers; Others.*

The last chapter „Posters“ has 93 papers.

The book is a great source of a lot of interesting results in physics education and in the same time is a very valuable acquisition for any library.

I. Grosu

D.W. HEERMANN, **Computer Simulation Methods in Theoretical Physics.** Springer Verlag, Berlin–Heidelberg–New York–London–Paris–Tokyo, ISBN 3-540-16966-0, 1986, 148 pp.

The book represents a one-semester introductory modern course for final year undergraduate, or first year graduate students. Reading the book we felt that simulation is a new tool of investigation in physics. The first two chapters contain some preliminaries useful in the next Chapters.

Chapter 3, *Deterministic Methods*, investigates: molecular dynamics (microcanonical ensemble molecular dynamics; canonical ensemble molecular dynamics; isothermal-isobaric

ensemble molecular dynamics; problems). The molecular dynamics method computes phase space trajectories of a collection of molecules which individually obey classical laws of motion. The Lennard-Jones potential of interaction is used.

Chapter 4, *Stochastic Methods*, treats: brownian dynamics, Monte Carlo methods (micro-canonical ensemble Monte Carlo method; canonical ensemble Monte Carlo method; isothermal-isobaric ensemble Monte Carlo method; ground ensemble Monte Carlo method; problems).

The book contains 2 appendices (*Random Number Generators and Program Listings*). The programs listed at the end of the book is of great aid to the users.

The book is a very important up-to-date information and practical book useful for students, researchers and professors of physics aiming to modernize their courses.

I. Grosu

K. SHIMODA, T. IYU (Eds.), *International Conference on Trends in Physics Education*, Sophia University, Tokyo, August 21-28, 1986. Published by The Physics Education Society of Japan, The International Commission of Physics Education (ICPE), Science Publishers, 1986, 182 pp.

The book begins with address to the International Conference on Trends in Physics Education by H. Kato (Nobel Prize). He suggests that new terms like interdisciplinary, nonlinear dynamics and fractals must be considered by physics education. The next address is by S. M. Iliadis (TIFCCO) who recommends "making physics more enjoyable".

The Chapter "Primary Lectures" contains 15 lectures given by leading personalities in research and development of physics education as follows:

A. Research in Physics Education and its Implementation in the Classroom, by H. Ullrich (Frankfurt); A. T. Liberman (France); Y. K. Kikuchi (Japan); K. T. O. (Singapore); P. M. (England).

B. The Impact of Developments in Technology on Classroom Teaching, by J. O. (Japan); H. T. (Japan); H. T. (Japan); Y. K. (Japan); K. M. (Japan).

C. The Motivation of Students Towards the Study of Physics, by V. T. (Japan); G. M. (Japan); E. L. (Japan); M. K. (Japan).

D. Special Lectures, by M. K. (Japan); H. K. (Japan); K. M. (Japan). These lectures contain a lot of important results obtained from research in physics education.

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I. Grosu

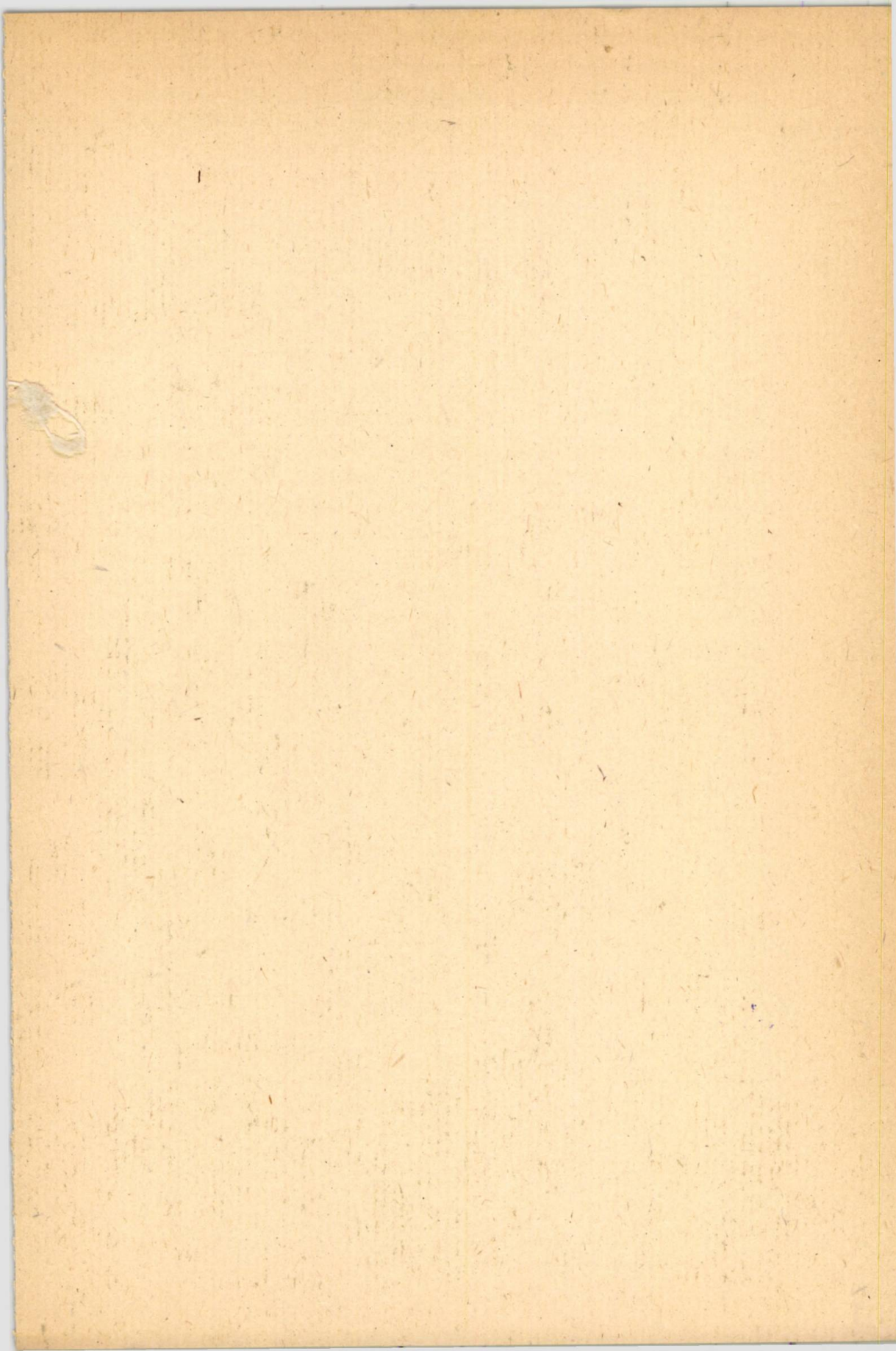
D. W. HERRMAN, *Computer Simulation Methods in Theoretical Physics*, Springer Verlag, Berlin Heidelberg New York London Paris Tokyo, 1986, 340 pp.

The book represents a one-semester introductory modern course for first year undergraduate or first year graduate students. During the book we will find simulation is a new tool of investigation in physics. The first two chapters contain some preliminary material in the next chapters.

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